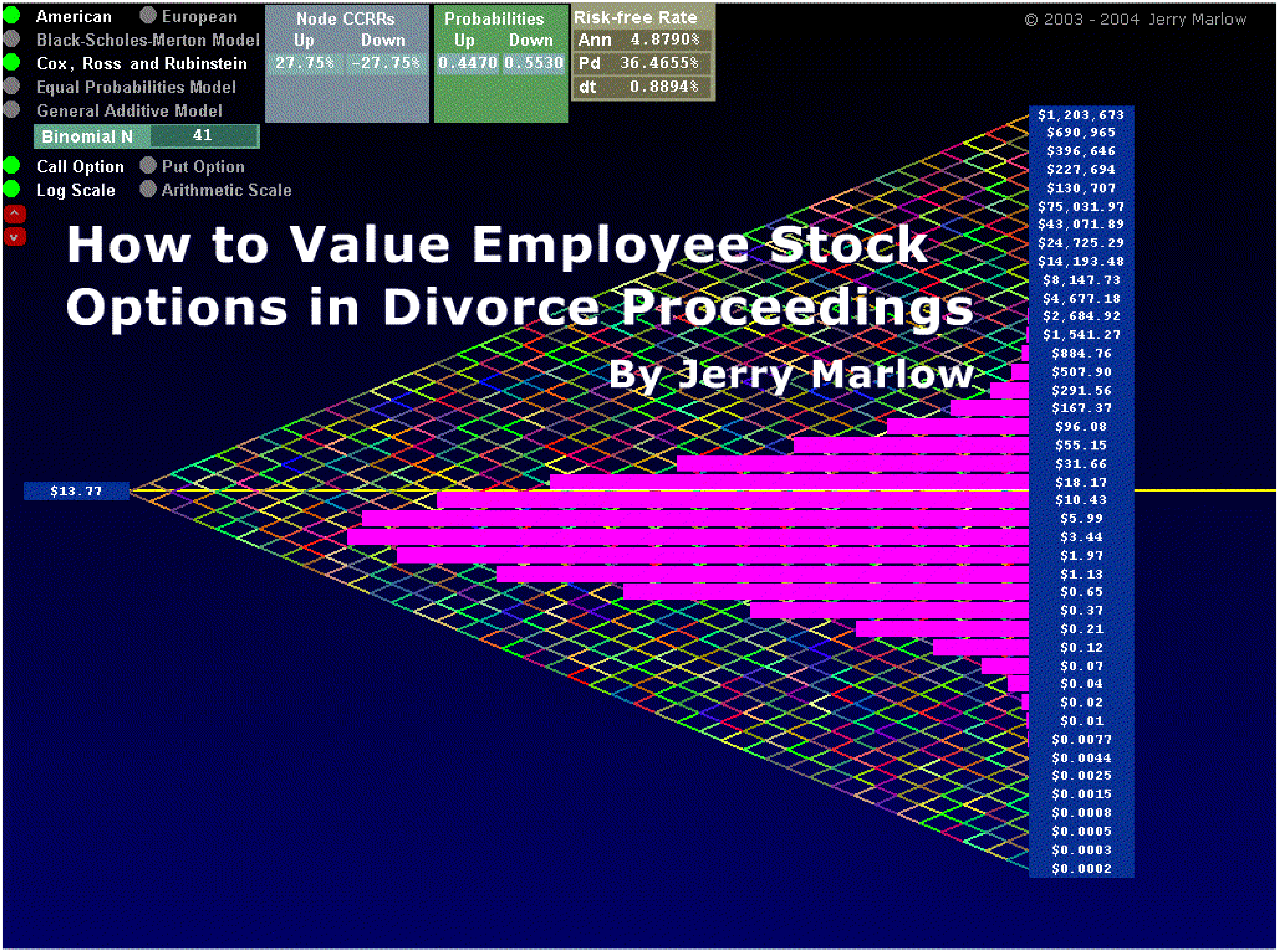
How to Value Employee Stock Options in Divorce Proceedings

By Jerry Marlow



Seminar Notes: How to Value Stock Options in Divorce Proceedings

By Jerry Marlow, MBA

Jerry Marlow values stock options in divorce proceedings and gives tutorials and seminars on how to value stock options in divorce proceedings. He is author of *Option Pricing: Black-Scholes Made Easy* published by John Wiley & Sons, Inc.

Jerry also gives tutorials and seminars on stock options, Black-Scholes option pricing theory, binomial option pricing theory and investment theory. He builds computer simulators of mainstream financial methodologies and writes educational and marketing materials about financial services and other complex issues.

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In many divorces today, one spouse's employee stock options account for a significant portion of the marital wealth to be divided. Even so, courts have yet to establish or recognize standard and reliable procedures with which to value these assets.

Valuing employee stock options in divorce proceedings presents a number of difficulties: In financial professions, Black-Scholes and risk-neutral methodologies are widely used and accepted for valuing market-traded stock options, but— as usually explained— these methodologies can be difficult for the non-mathematician to understand. Depending on the factual context of a particular set of employee stock options, these methodologies may be applicable, may not be applicable, or may be applicable only as a starting point for the valuation of the options.

To add to the potential for confusion, what are commonly referred to as option grants often— in two respects— do not in fact grant options: What the grants define as options may not conform with the definition of a market-traded option on which accepted valuation methodologies rely. Even what the grants themselves define as options they may not actually grant to the employee at the time of the so-called grant.

Interspersed with some words being used to mean things other than what they mean, other words may carry different implications within the legal and financial professions. In the legal realm, if an attorney successfully lambastes a financial prospect as speculative or as an expectancy, a court may be inclined to regard the prospect as being of zero dollar value. By contrast, in the financial realm, a speculator is defined as someone who expects to profit by taking on exposures to risk. In the financial realm, every forecast is a probability distribution. An expected return is defined as the probability-weighted average return of a given forecast. While speculative ventures and expectancies may get thrown out of court, without speculators and expected returns the financial markets might grind to a halt.

Of a perhaps more esoteric but nonetheless real concern is whether and how courts regard risk in their valuations of employee stock options. The essence of stock options is *uncertainty* about the future market price of the underlying stock on which the option is written. Valuing options requires moving uncertain values across time. Accordingly, under the standard valuation methodologies, the value of an option is a probability-weighted present value. In a sense, the probability-weighted present values used to value options ignore how risk averse an employee may be to having a substantial portion of his or her wealth concentrated in an asset of highly uncertain future value. Does a high likelihood that the employee spouse may reap no payoff from the options somehow—rationally or psychologically— diminish the legitimacy of a probabilistic approach to valuing them?

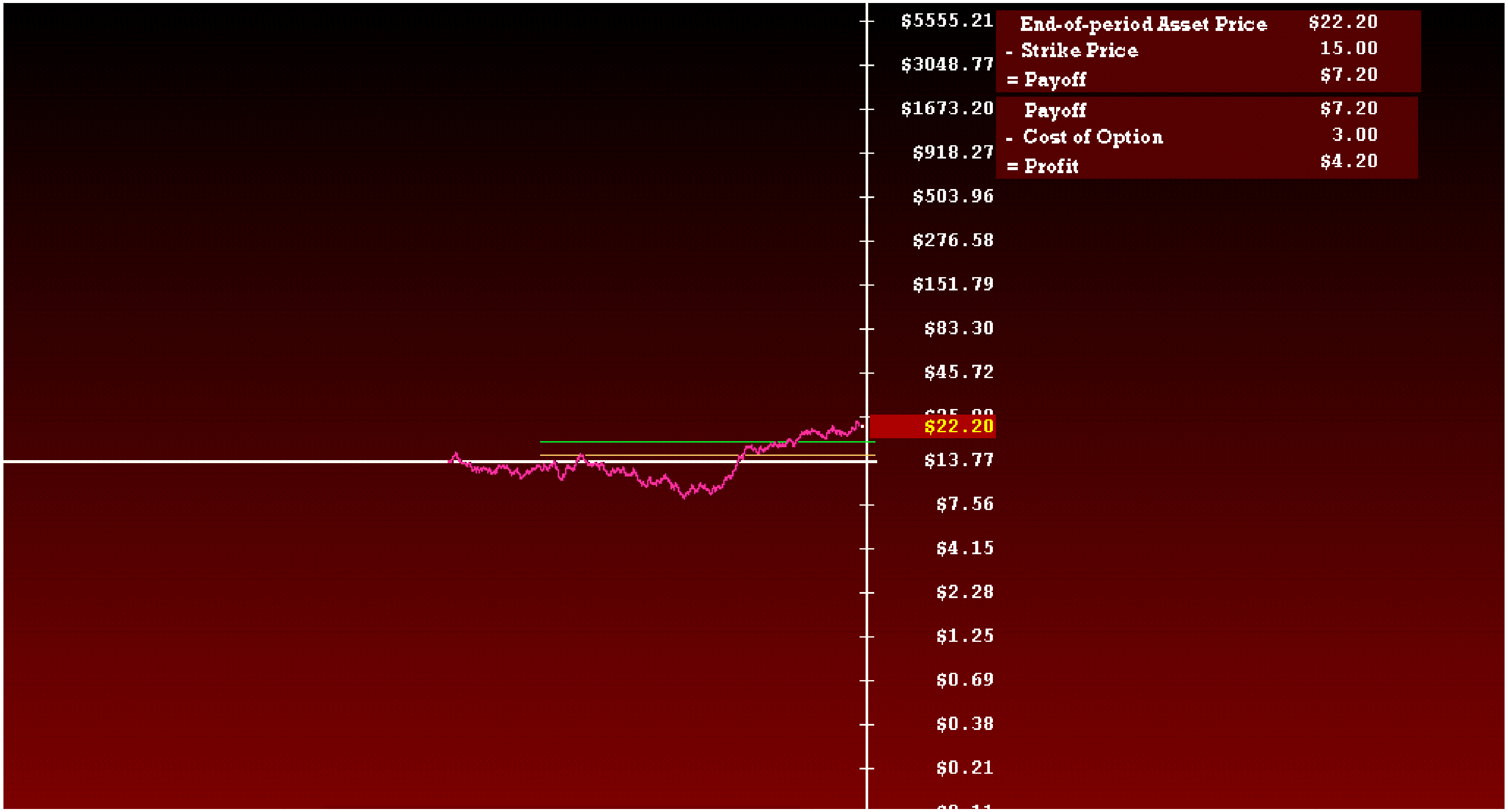
Even if a judge, the spouses' attorneys and their respective expert witnesses were working collaboratively to arrive at a fair valuation of employee stock options, the task could be daunting. Thrown into the adversarial process, the task has the potential to prove maddening to all.

To make it easier for judges, attorneys, valuation professionals and divorce financial planners to arrive at fair valuations of employee stock options, I here offer a conceptual context for thinking about the fair value of options and a sequence of steps for calculating that value. To begin, we look at how the potential payoffs of market-traded options translate into option values. Then we look at ways to apply these principles to employee stock options. Our goal is to calculate values for employee stock options that are consistent with the values of market-traded options and that compensate the employee spouse for his or her continued exposure to uncertainty.

An option's payoff is uncertain.

Up until its expiration, a market-traded call option gives its owner the right to buy the underlying stock at a stated strike price. If, during that time, the market price of the underlying stock goes above the strike price, the owner of the option can buy the stock at the strike price and sell it at the market price. The payoff is equal to the difference between the market price and the strike price. Hence the payoff of the option depends on where the market price of the underlying stock goes before the option expires.

To illustrate, let's simulate possible payoffs of a call option on XYZ stock, which is currently trading at $13.77. An investor buys an XYZ call option with a strike price of $15.00. The option costs $3.00. It expires in 365 days.

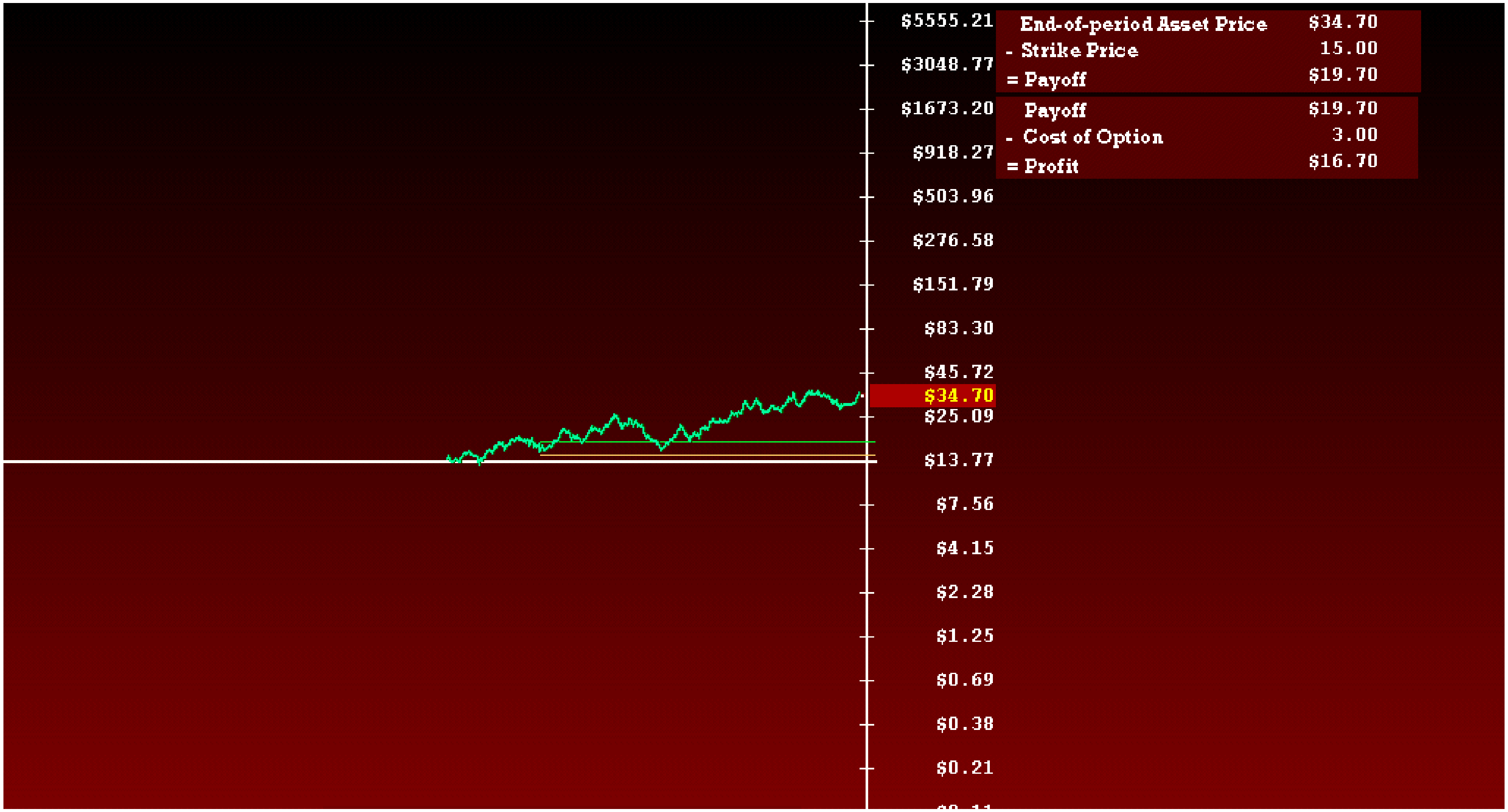


In the simulation, the yellow line at $15.00 represents the strike price. For the option to be in the money when it expires, the stock price must be above the yellow line.

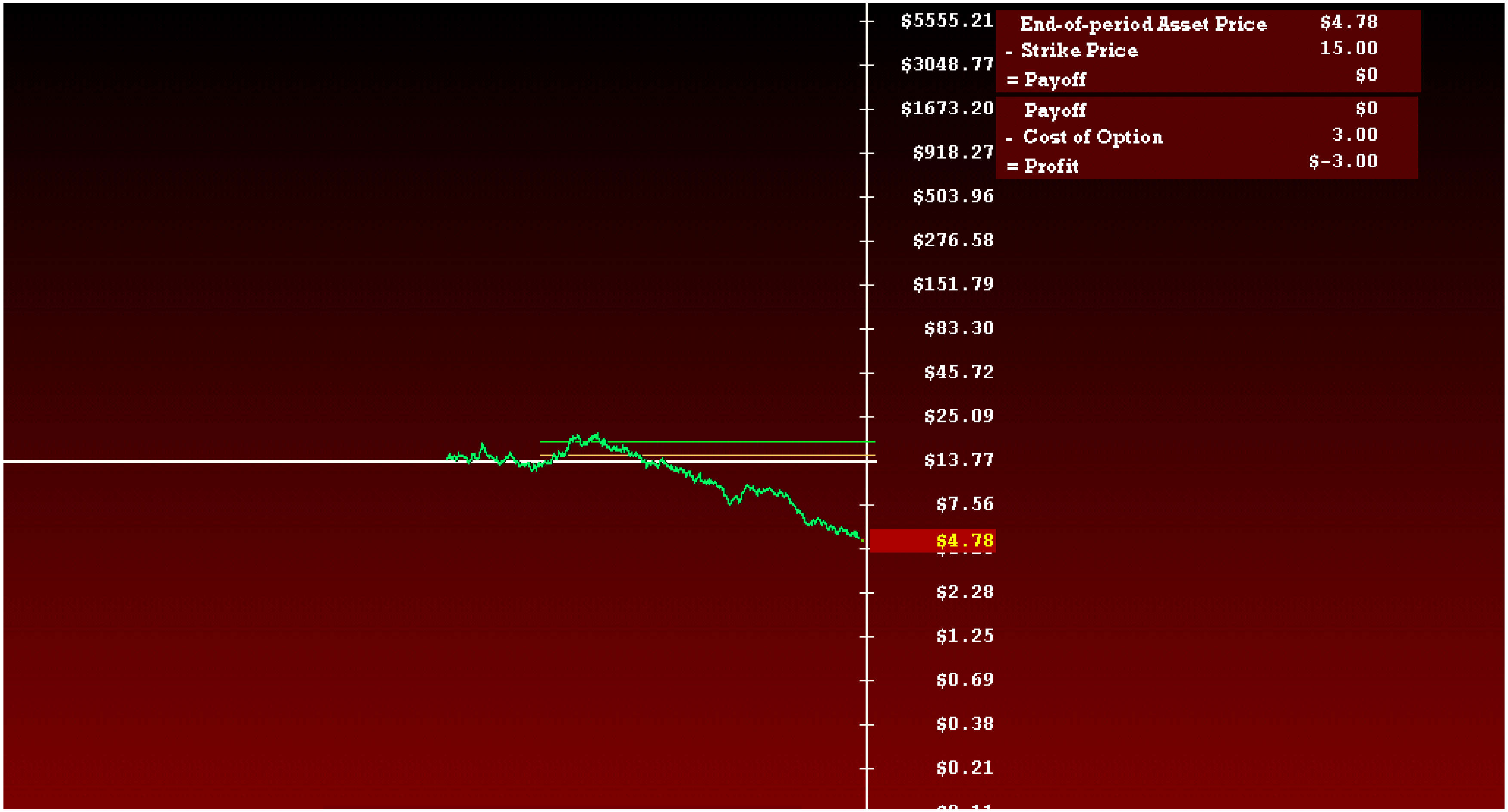
The distance from the yellow line to the green line represents the cost of the option. To produce a profit, when the option expires, the stock price must be above the green line.

In this simulation, over the course of the next 365 days, the stock price goes from $13.77 to $22.20. The end-of-period stock price minus the strike price of $15.00 gives a payoff of $7.20.

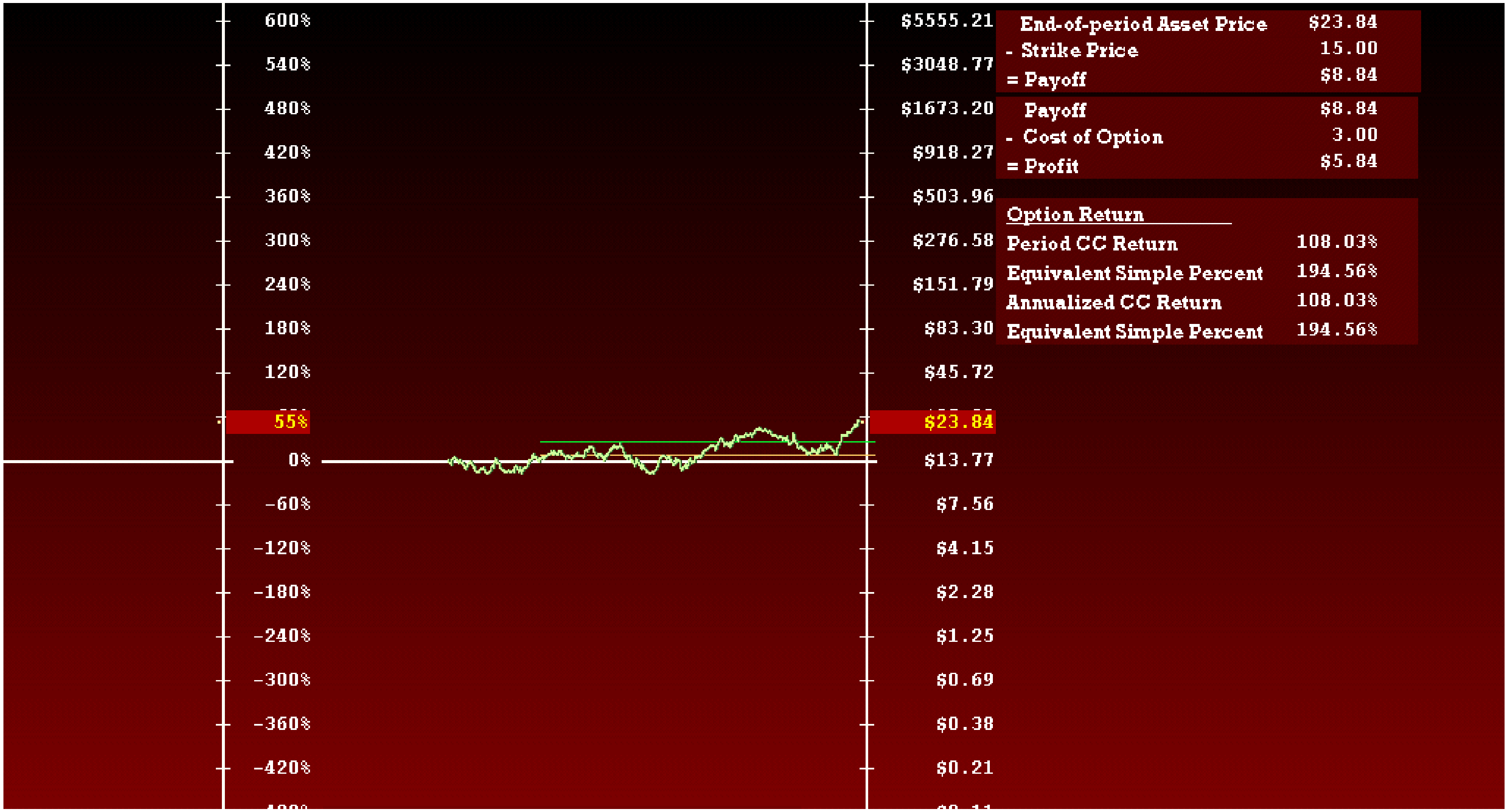
The payoff minus the $3.00 cost of the option gives the investor a $4.20 profit.



If, instead, over the next 365 days, the price of XYZ stock goes from $13.77 to $34.70, then the payoff on the option would be $19.70. The profit would be $16.70.

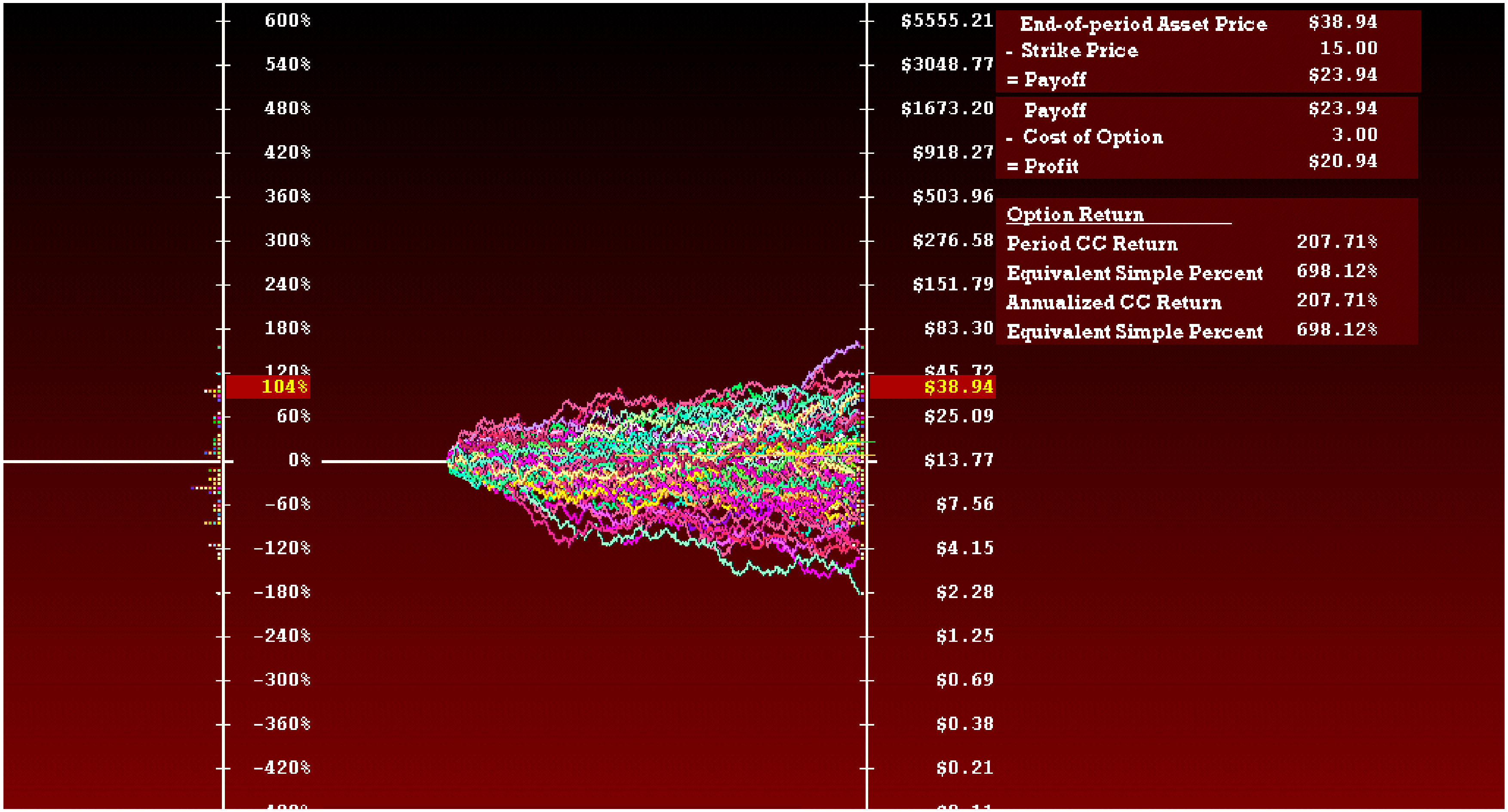


On the other hand, if, over the next 365 days, the price of XYZ stock goes from $13.77 to $4.78, then the option finishes out of the money. The payoff is zero. The investor has a loss equal to the cost of the option.



In addition to simulating price paths, we can record the stock returns that the price paths produce. A price change from $13.77 to $23.84 would produce a continuously compounded rate of return of 55%. In the simulation, we tabulate the return with a little square at the height of 55% on the return axis.

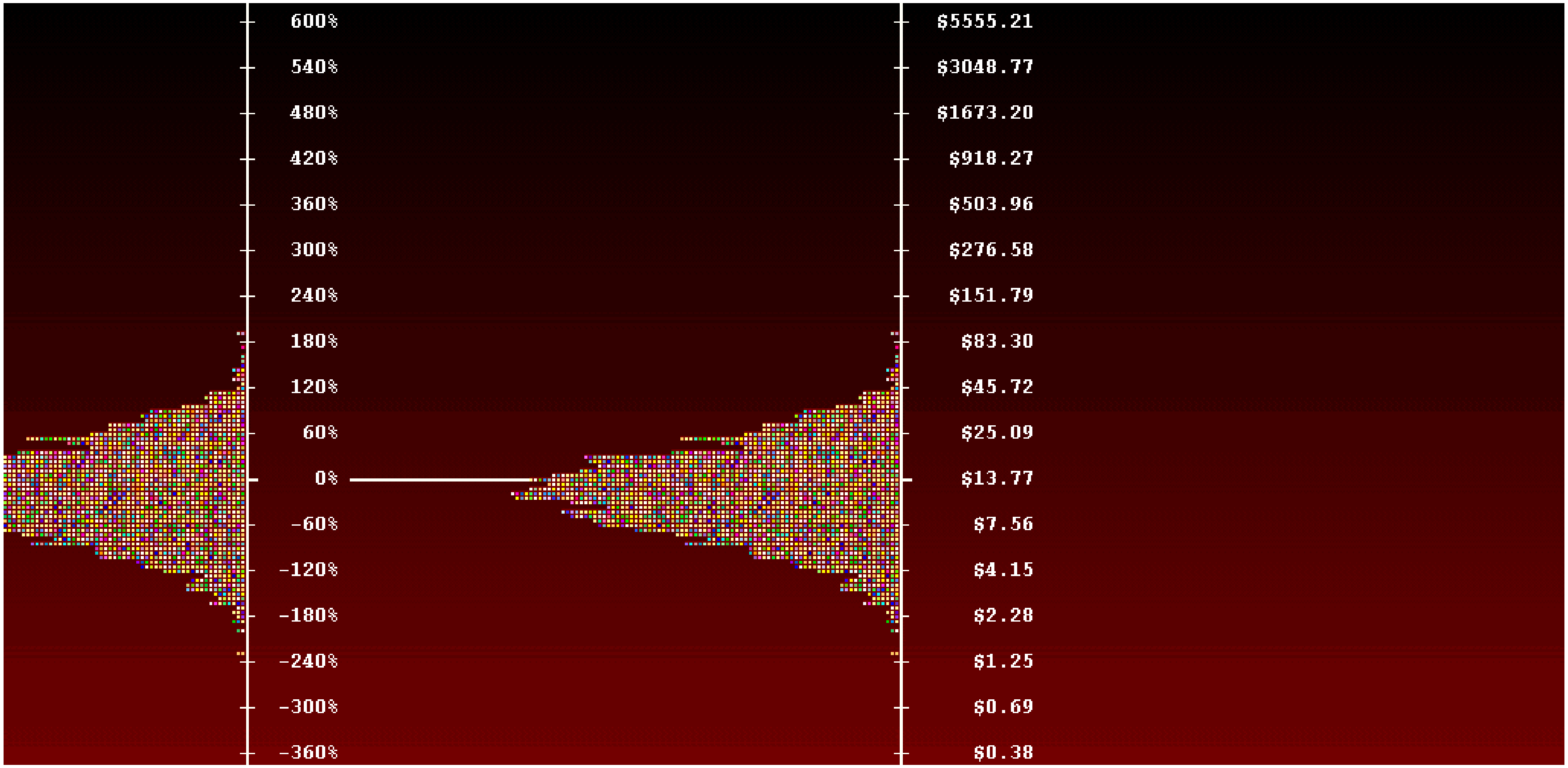
We also can calculate the rate of return that investing in the stock option produces. A profit of $5.84 on an investment of $3.00 would be a continuously compounded rate of return of 108.03%—which is equivalent to a simple percentage return of 194.56%.



An option's value is related to its potential payoffs and the probability of those payoffs.

We can simulate any number of possible paths that a stock price *might* follow. Each possible price path would produce a payoff. If the final price is above the strike price, the payoff will be positive. If the final price is below the strike price, the payoff will be zero.

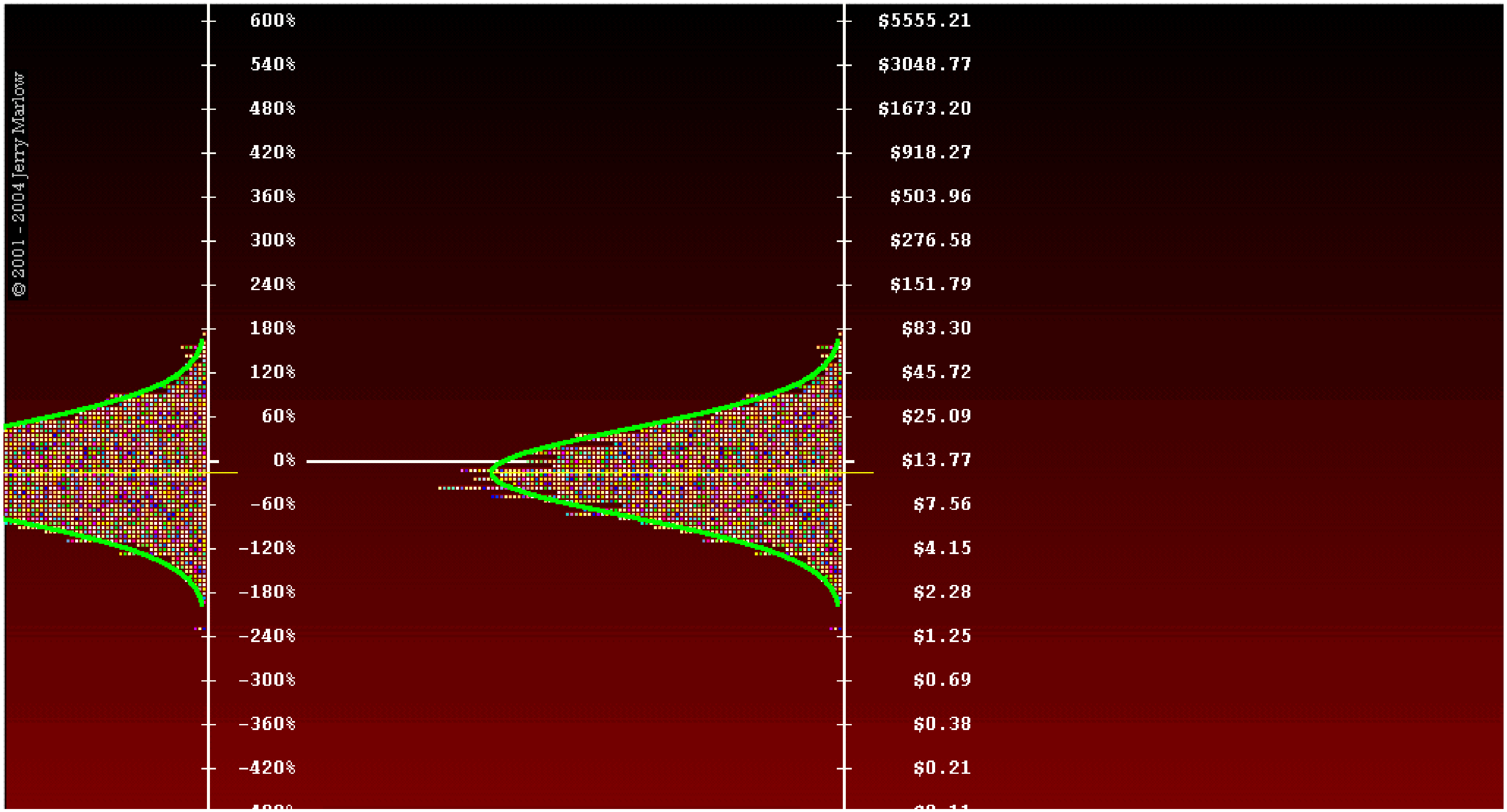
An option's value is related to its potential payoffs and the probability of those payoffs. How can we evaluate all the potential payoffs and their probabilities in a systematic way?



If we tabulate the outcomes of thousands of price-path simulations with little squares, the squares form a pattern.

To start, instead of drawing the entire price path for each simulation, we can tabulate the outcome of each simulation with a little square.

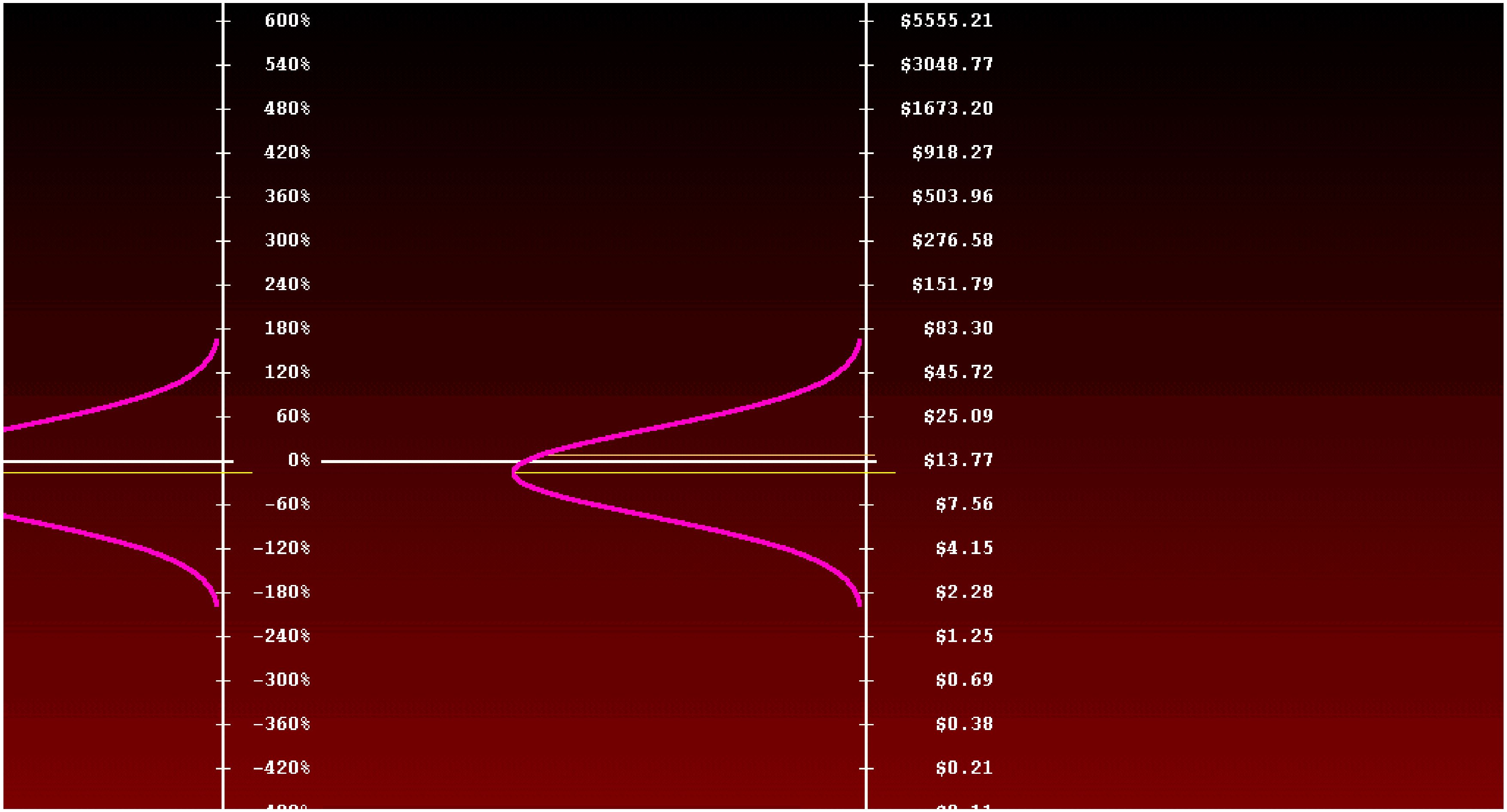
When we run a couple of thousand price-path simulations and tabulate each outcome with a little square, we see that the little squares form a pattern.



Every financial forecast is a probability distribution. A stock forecast is a bell-shaped curve.

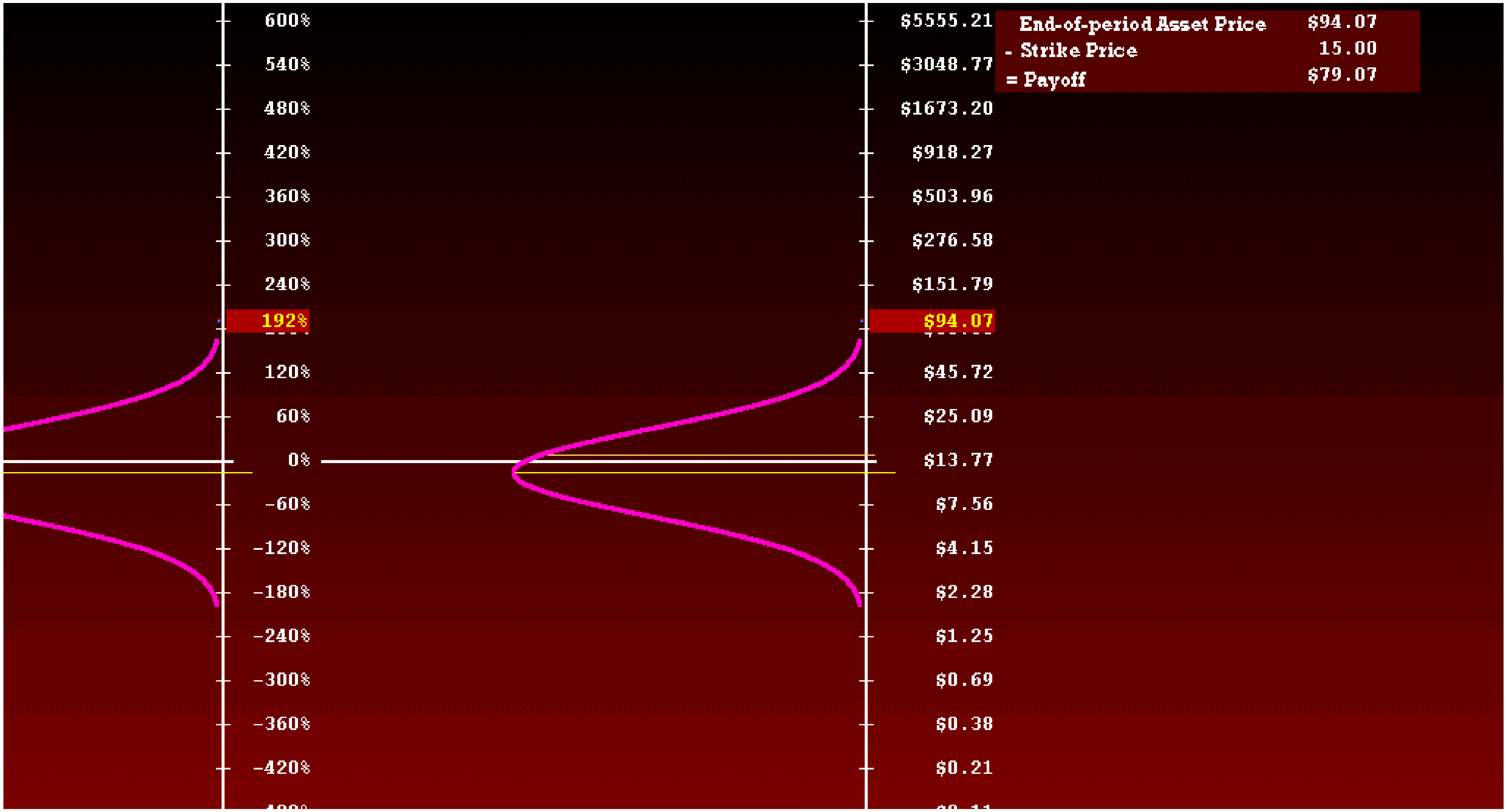
The pattern of prices at the end of the investment period approximates a bell-shaped curve—otherwise known as a normal distribution. Black-Scholes options pricing theory and other standard option-valuation methodologies assume that the likelihood of a stock's future price is distributed in this way.

Every financial forecast is a probability distribution. Graphically, a stock forecast is a bell-shaped curve. Every outcome in the bell-shaped curve produces an option payoff. Hence, one way we can calculate the value of an option is to find the probability-weighted present value of all the payoffs produced by the outcomes in the bell-shaped curve.

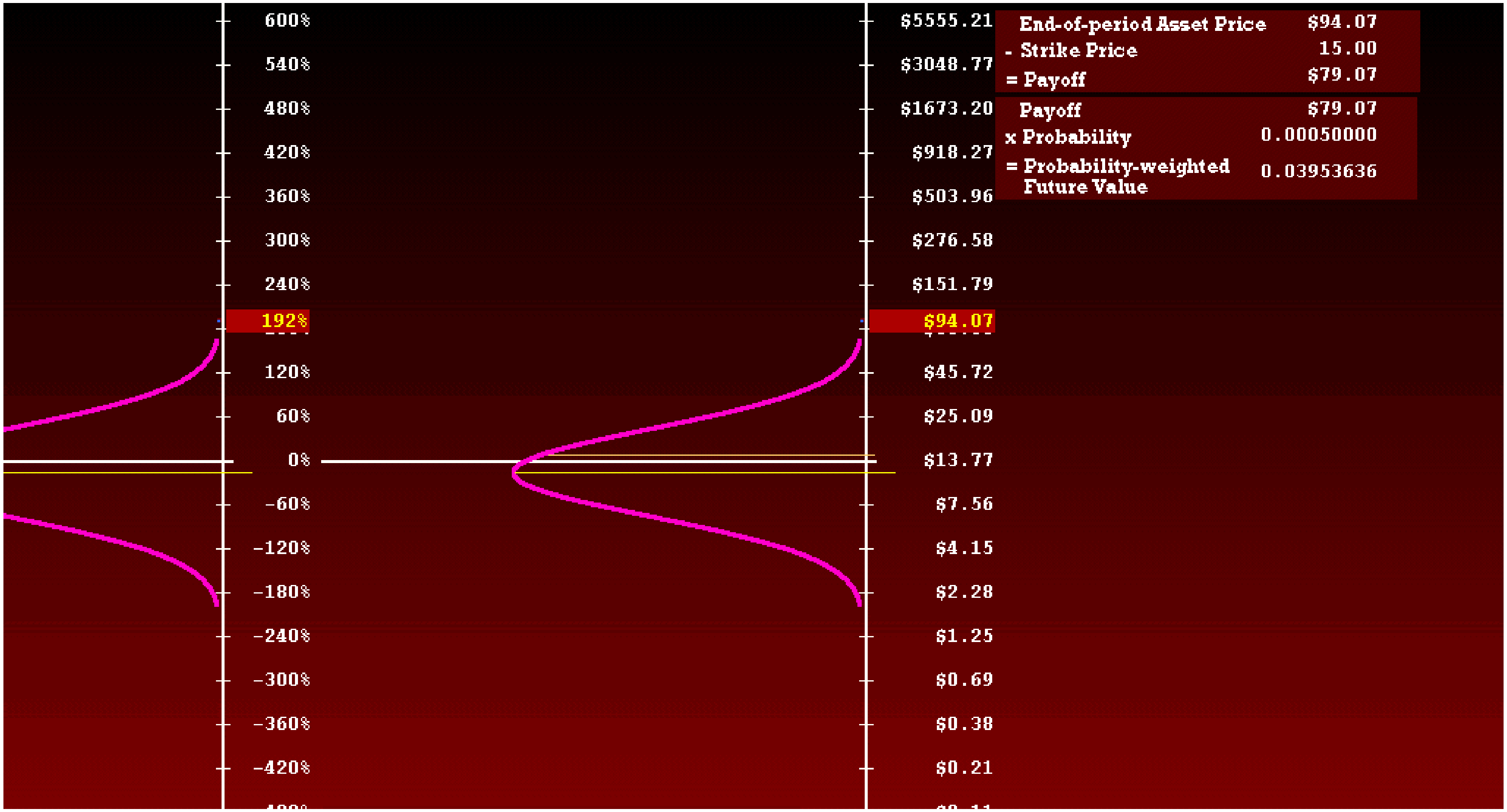


To calculate the value of an option, we can evaluate the payoffs produced by the squares that fill up the bell-shaped curve.

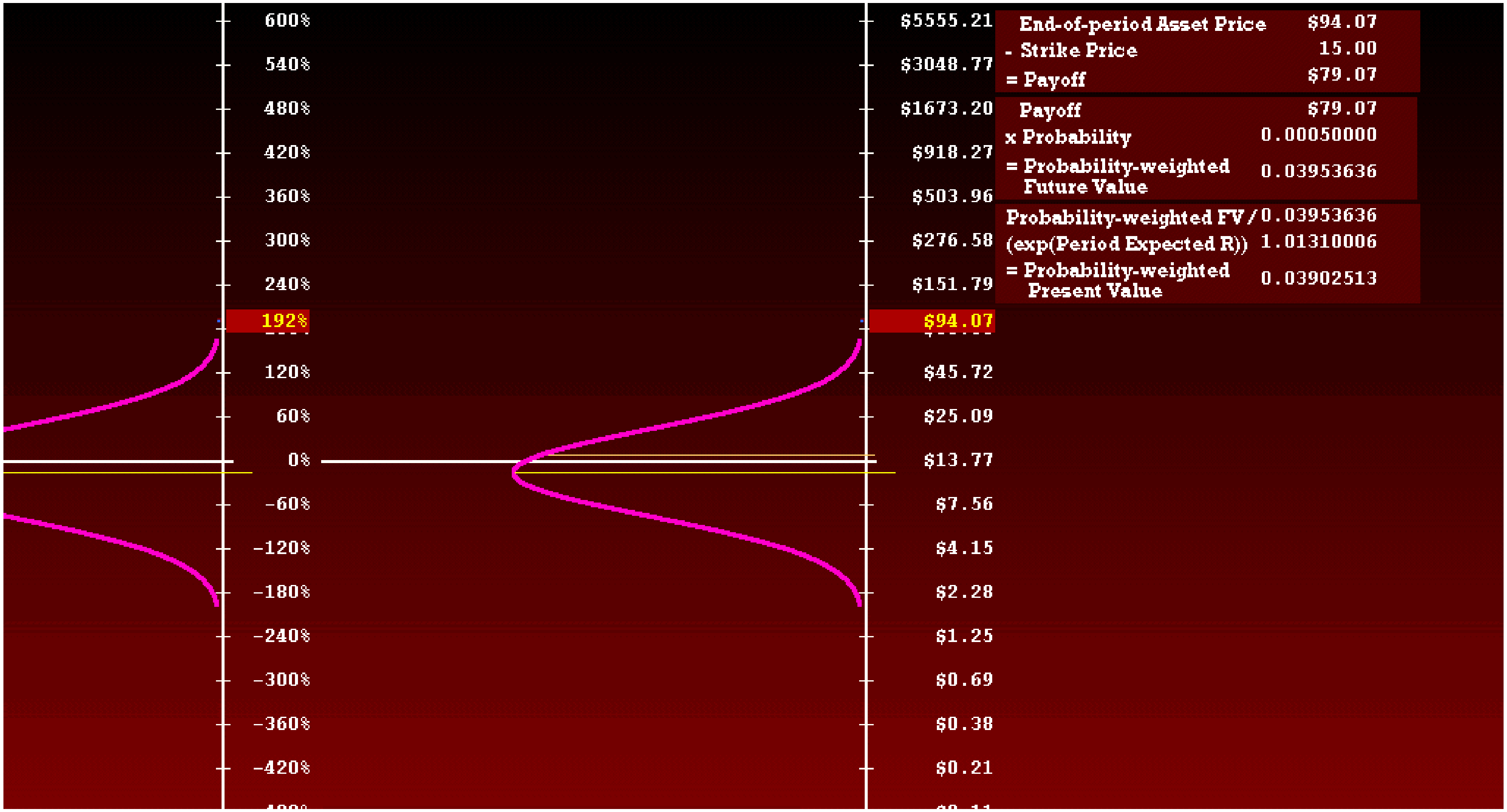
If we take this bell-shaped probability distribution to be a good representation of where the price of this stock might be 365 days from now, then we can calculate the probability-weighted present value of all the payoffs that would be produced by stock prices within and just beyond this bell-shaped curve.



If we divide the bell-shaped curve into 2,000 possible outcomes, then the highest stock price we might expect for this stock at the end of 365 days would be $94.07. This stock price would give us an option payoff of $79.07. We want to find the probability-weighted future value of this payoff.

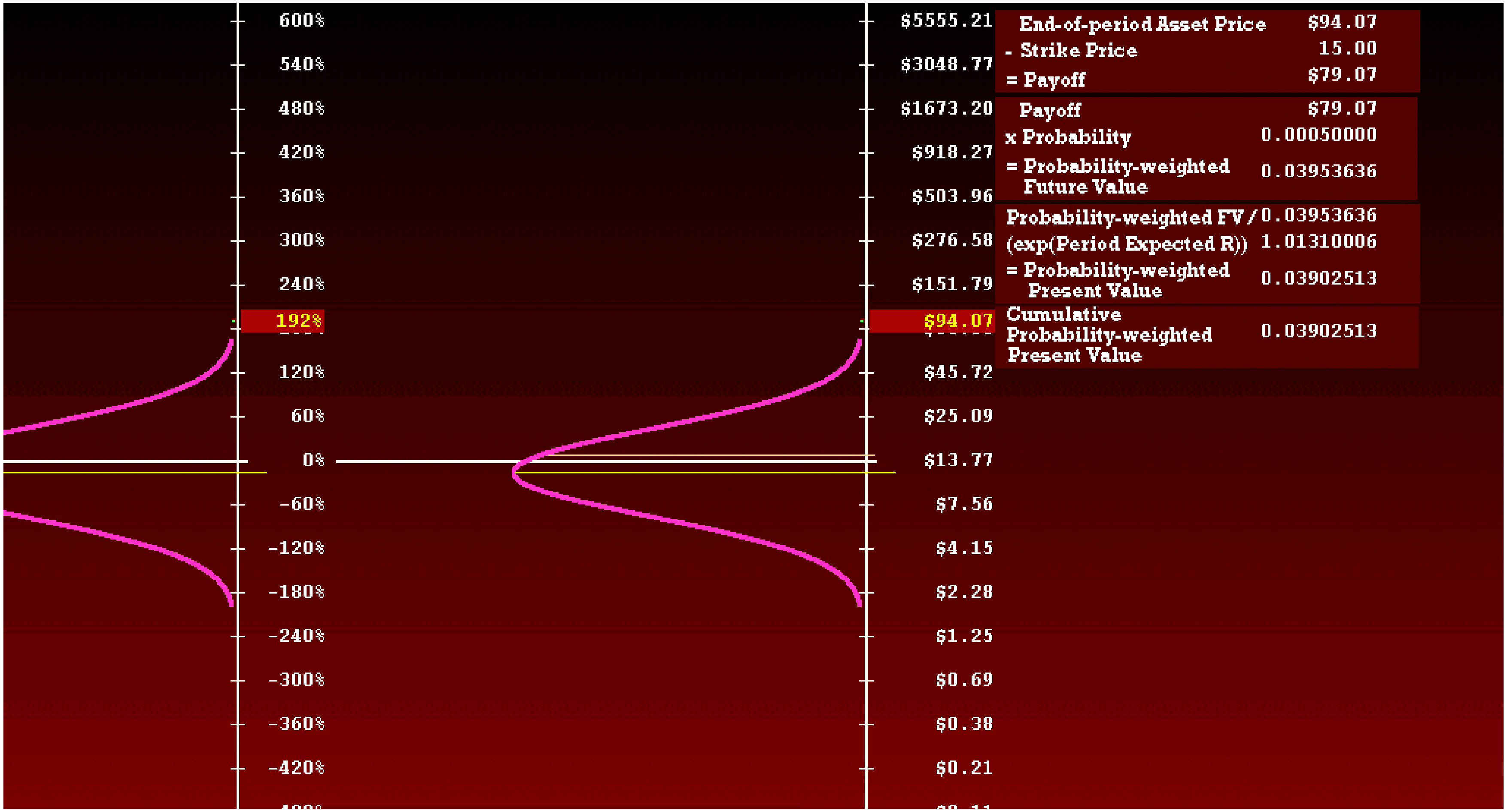


Because we're using 2,000 possible outcomes, the probability of this outcome is 1 / 2000 or .0005. Hence, the probability-weighted future value of this payoff is $79.07 x .0005 = $0.03953636, or roughly 4 cents.

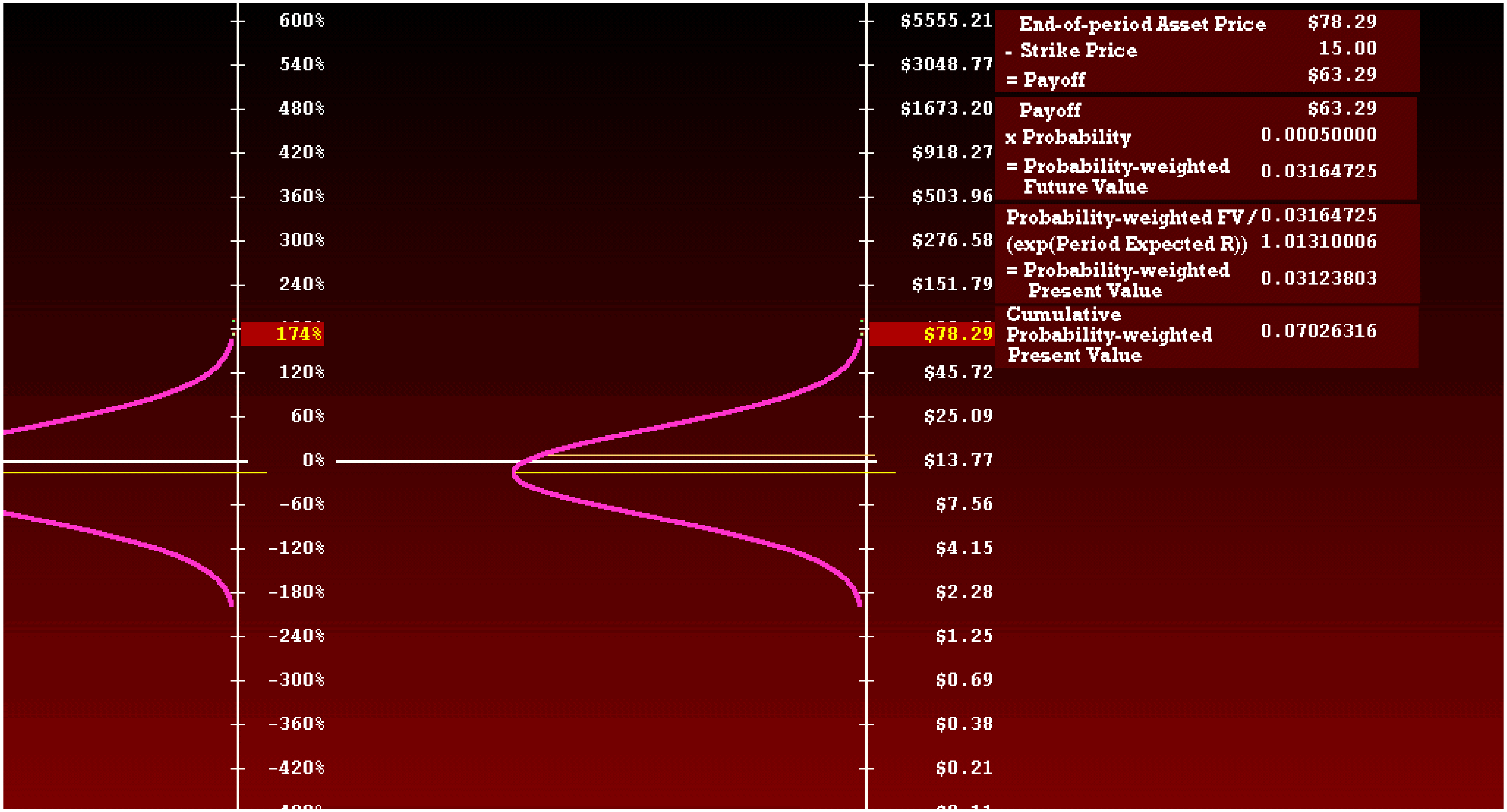


To find the value *today* of an option that can pay off *in the future*, at the risk-free rate of interest we discount the probability-weighted future values of the potential payoffs. To discount them, we divide the probability-weighted future values of each payoff by one plus the risk-free rate of interest for the option's time to expiration.

For this payoff, we divide its probability-weighted future value of $0.03953636 by 1.01310006. We get a probability-weighted present value of $.03902513— still right around 4 cents.

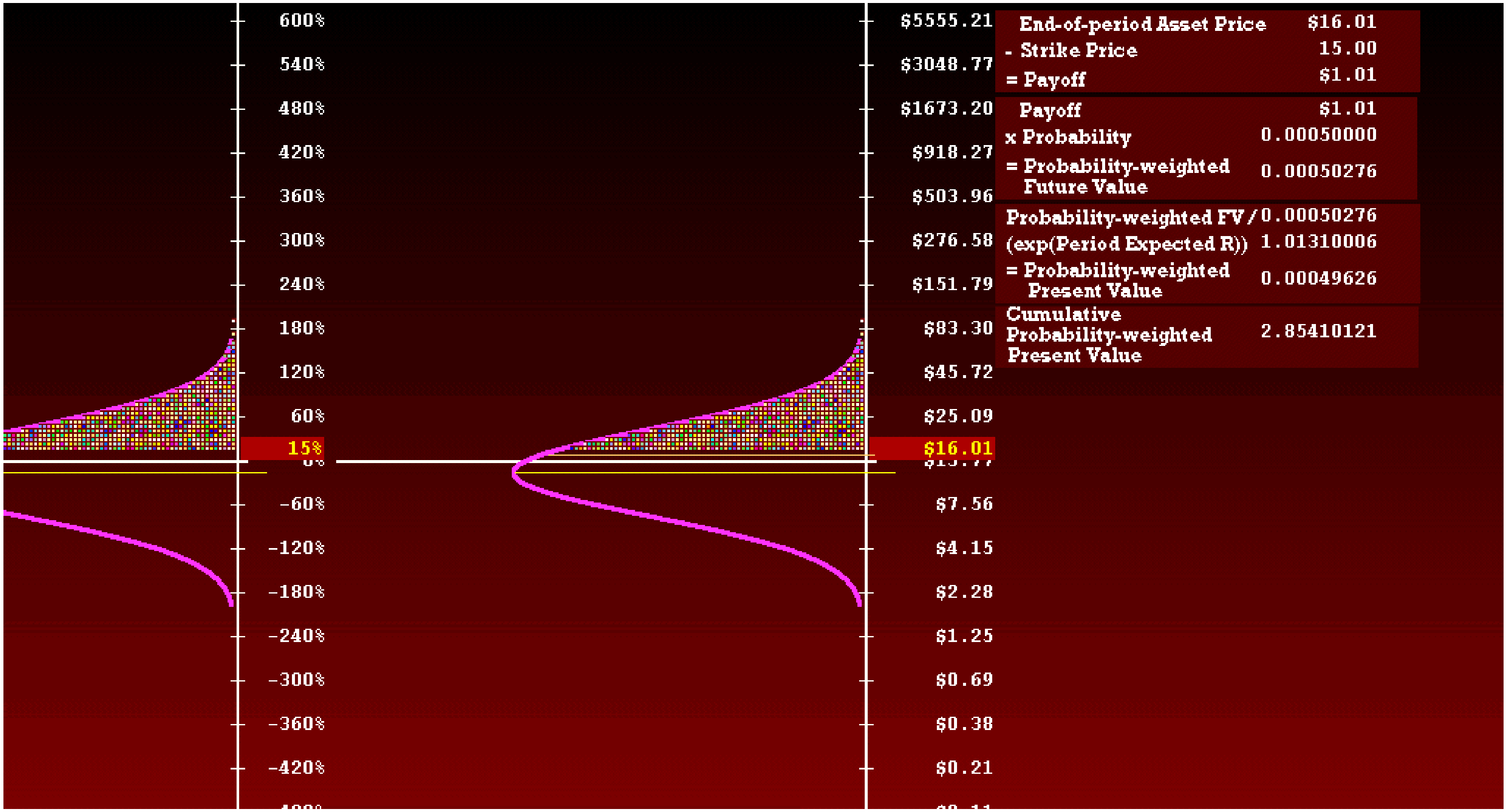


We've calculated the probability-weighted present value of this one payoff and we save it.



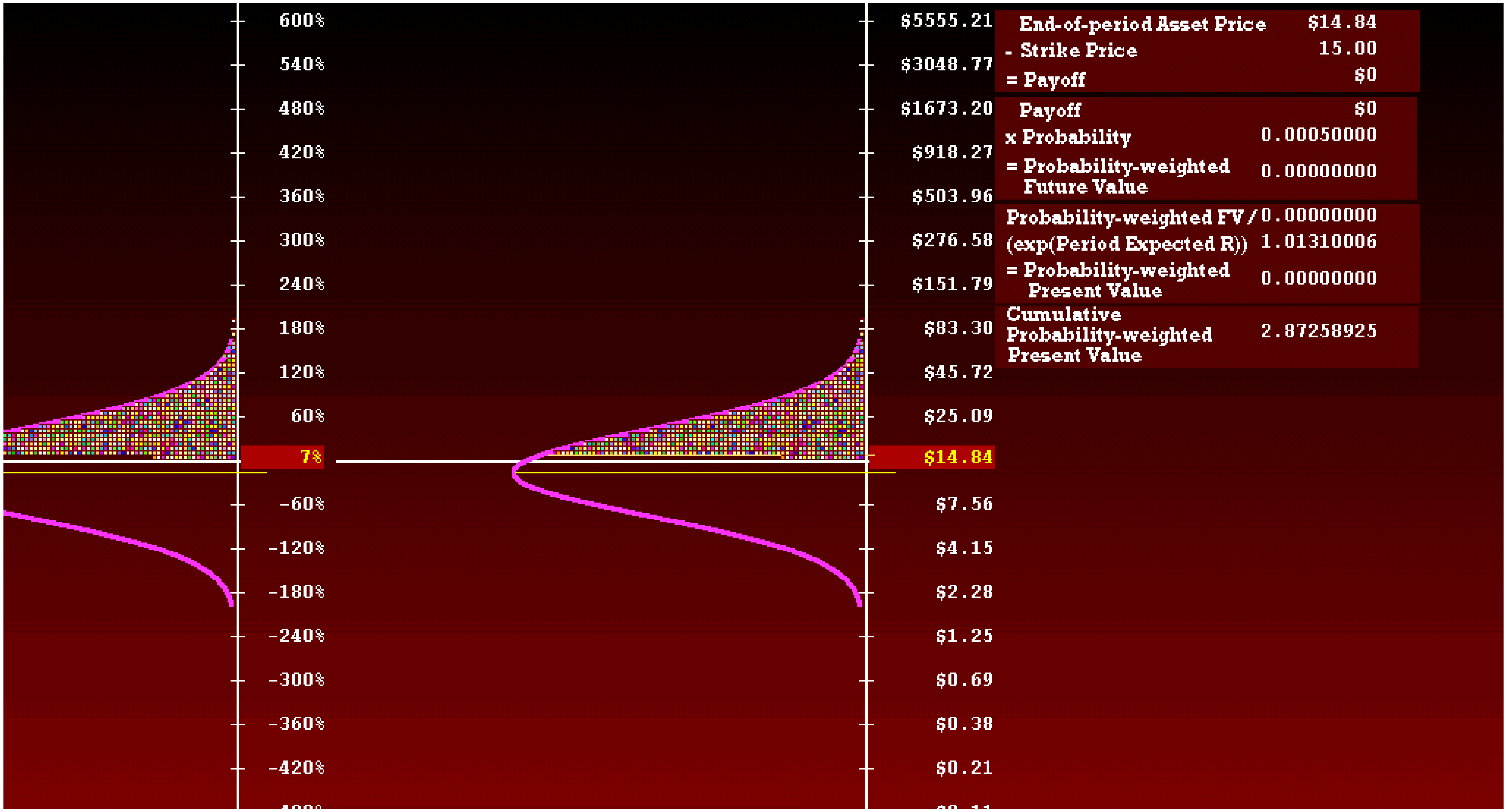
With the forecast we're using for this stock, the second highest price outcome we might expect at the end of 365 days is $78.29. Its payoff is $63.29. Its probability-weighted future value is $0.03164725. Its probability-weighted present value is $0.03123803. We add this payoff's probability-weighted present value to that of the first outcome.

And so we work our way through the entire bell-shaped probability distribution.



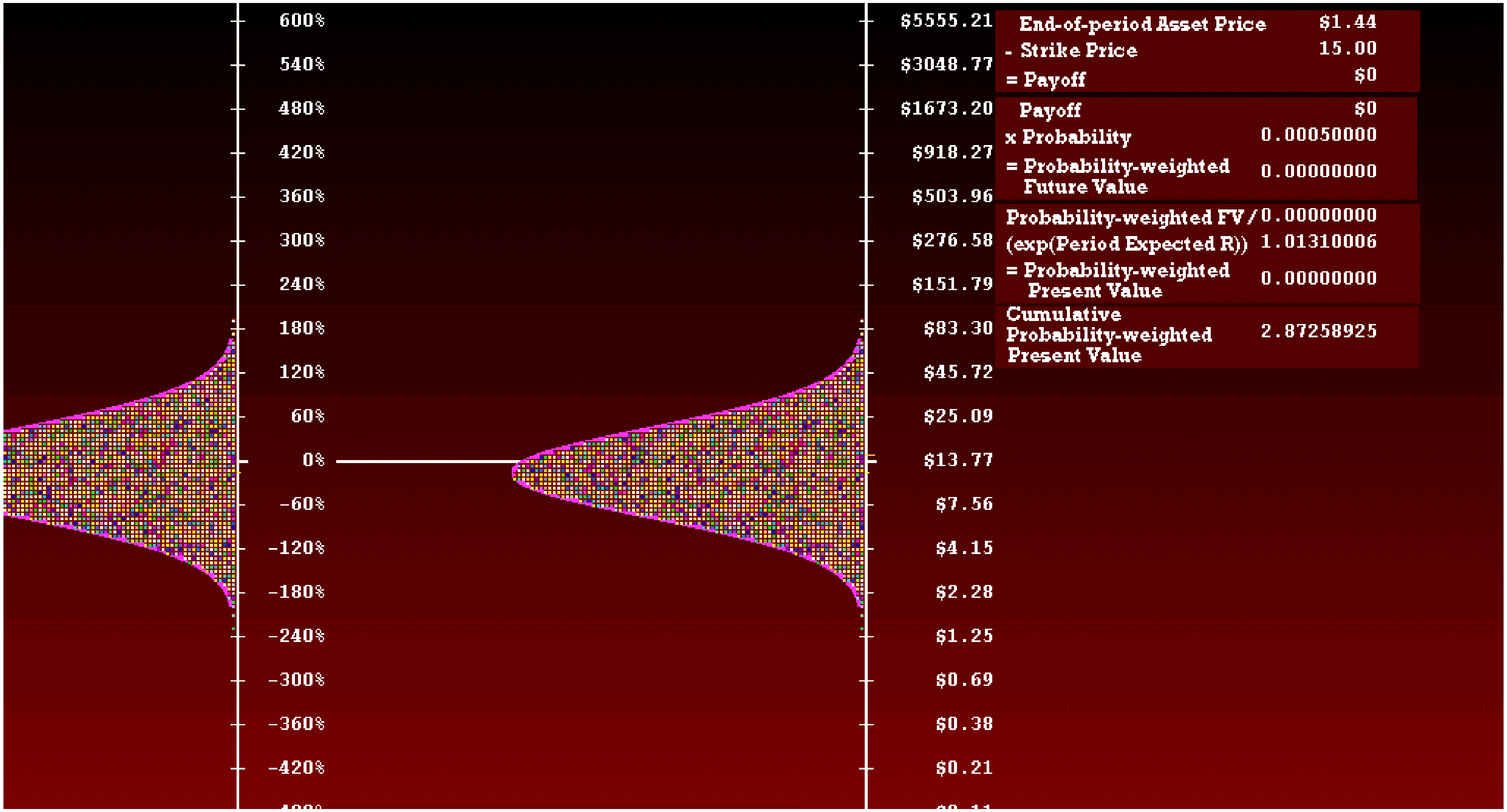
Payoffs near the strike price add little to the value of the option.

We see that, when the stock price at the end of the 365 days is close to the strike price, the payoff is small. It adds very little to the cumulative probability-weighted present value. A payoff of $1.01 adds less than a twentieth of a penny to the value of this option.



Payoffs below the strike price add nothing to the value of the option.

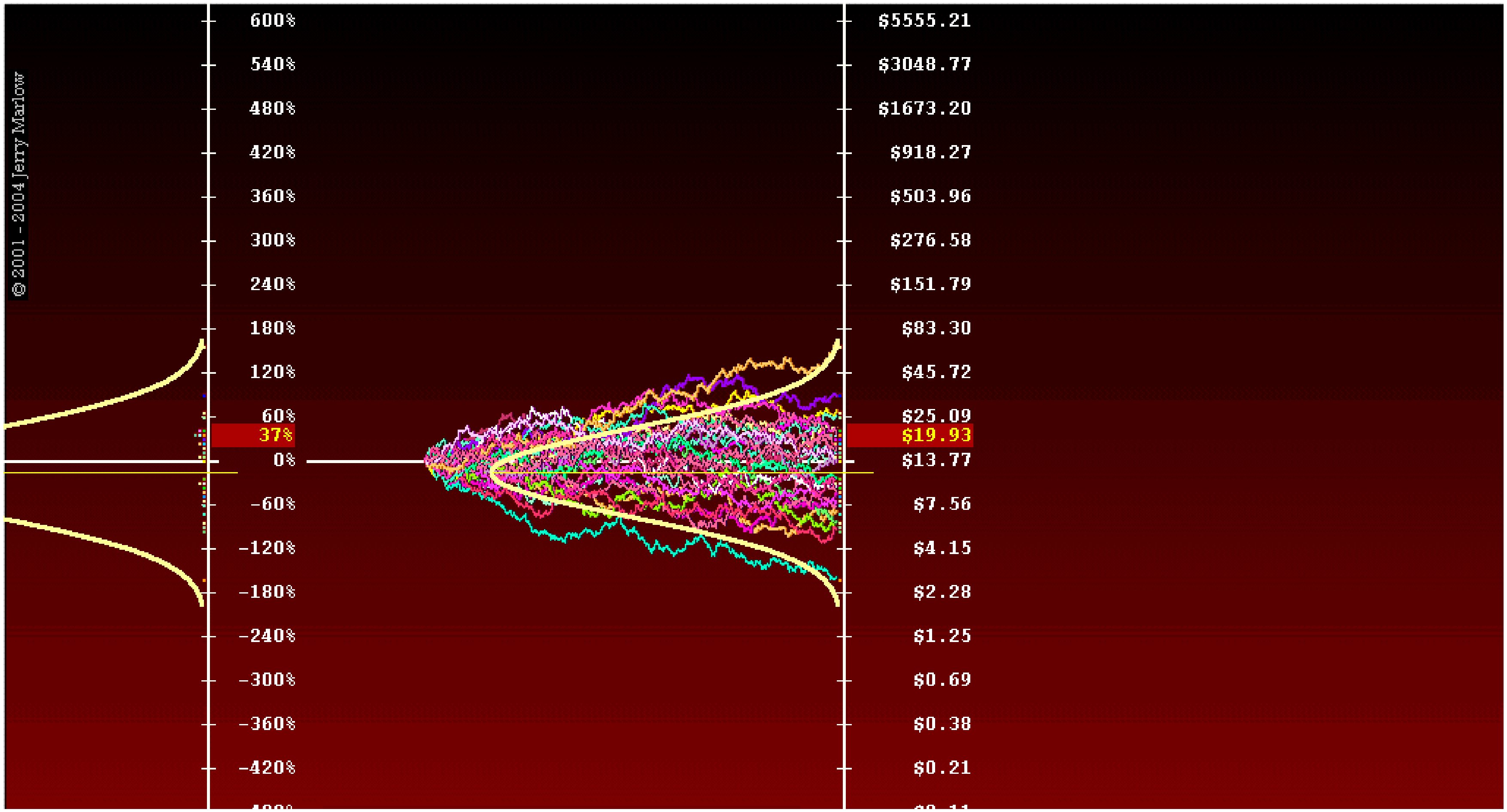
Once the stock price is below the option's strike price, the payoff is zero. It adds nothing to the option's probability-weighted present value.



We get an option value half way between the marketplace bid and ask prices.

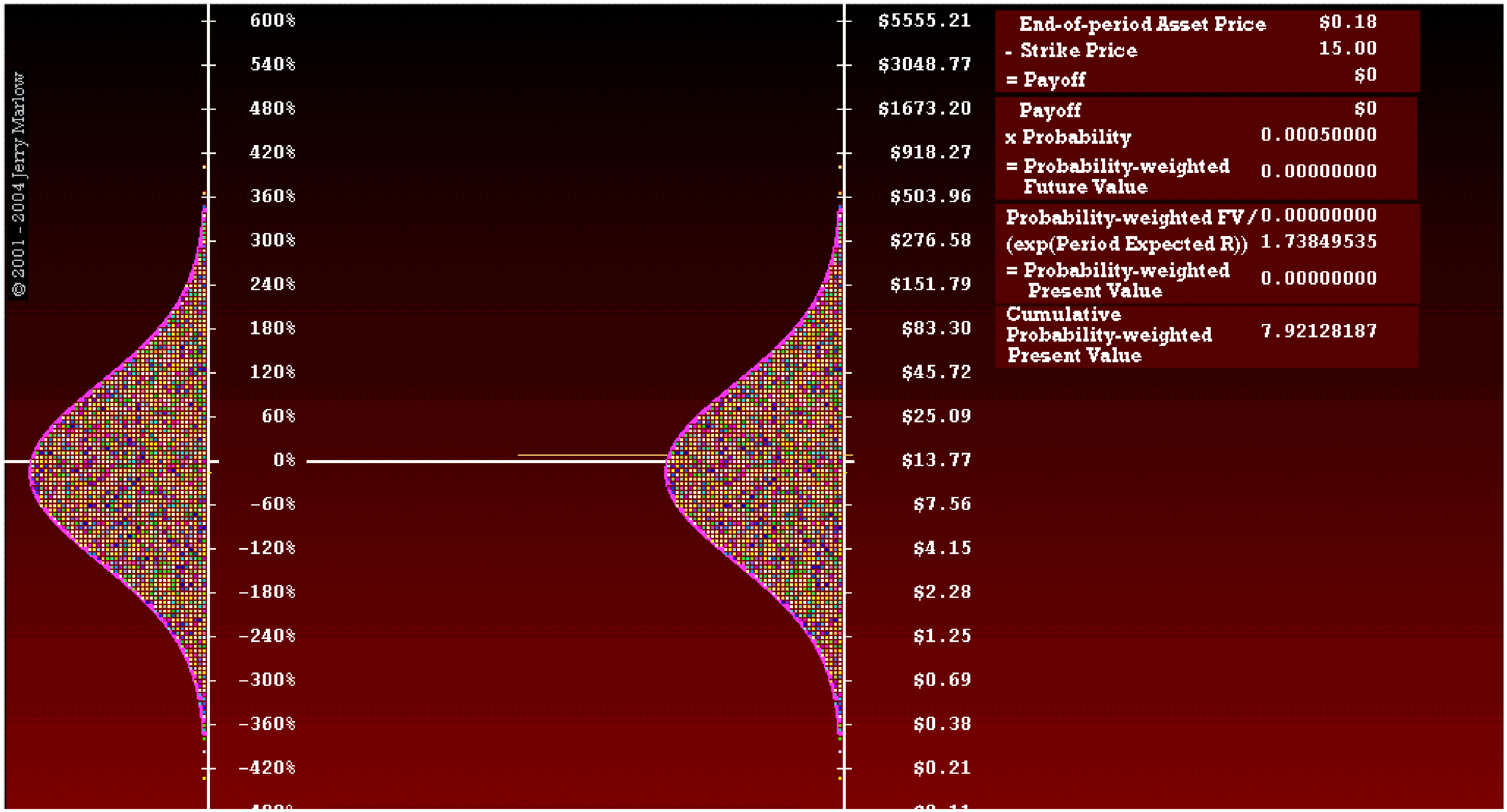
When we divide the bell-shaped curve into 2,000 possible outcomes, the option value we get is $2.87258925. Had we been more precise and divided the bell-shaped curve into 100,000 possible outcomes, we would've gotten an option value of $2.8754.

In the marketplace, the ask price for this option is $3.00. The bid price is $2.75. The value we calculated is half way between the bid and ask prices.



The spread of the bell-shaped curve and how high it sits on the price axis determines the option's value.

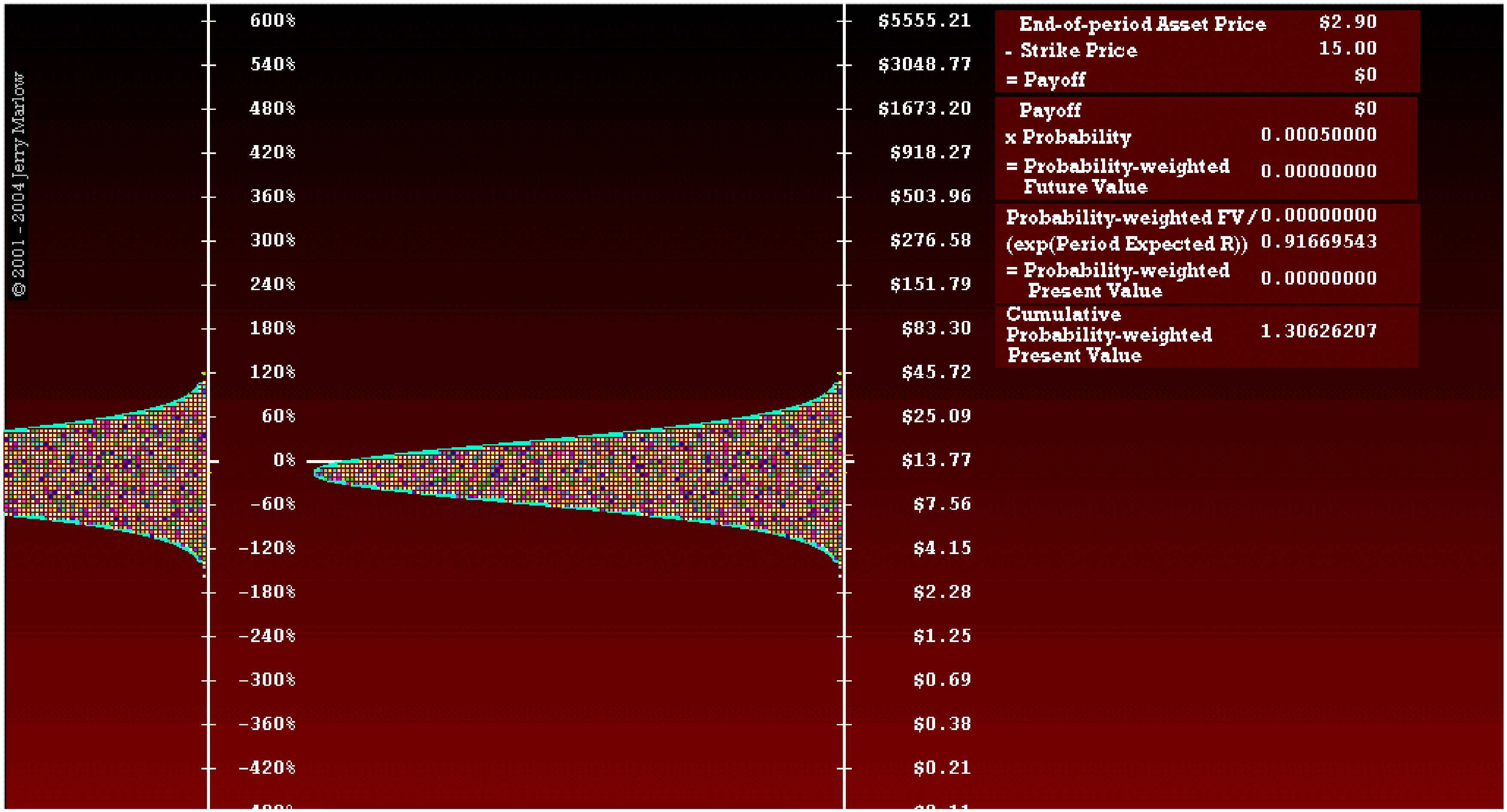
To value this option, we drew the stock forecast as a bell-shaped curve, filtered the curve through the option's strike price, and found the cumulative probability-weighted present value of the potential payoffs. Had the bell-shaped curve been higher, lower, more spread out or less spread out, the option would have had a different value. The spread of the bell-shaped curve and how high it sits on the price axis determines the option's value.



A more spread out bell-shaped curve centered at the same height gives a higher option value.

The stock forecast we evaluated previously gave an option value of $2.87. The bell-shaped curve shown here is centered at the same height on the stock-price axis but is more spread out.

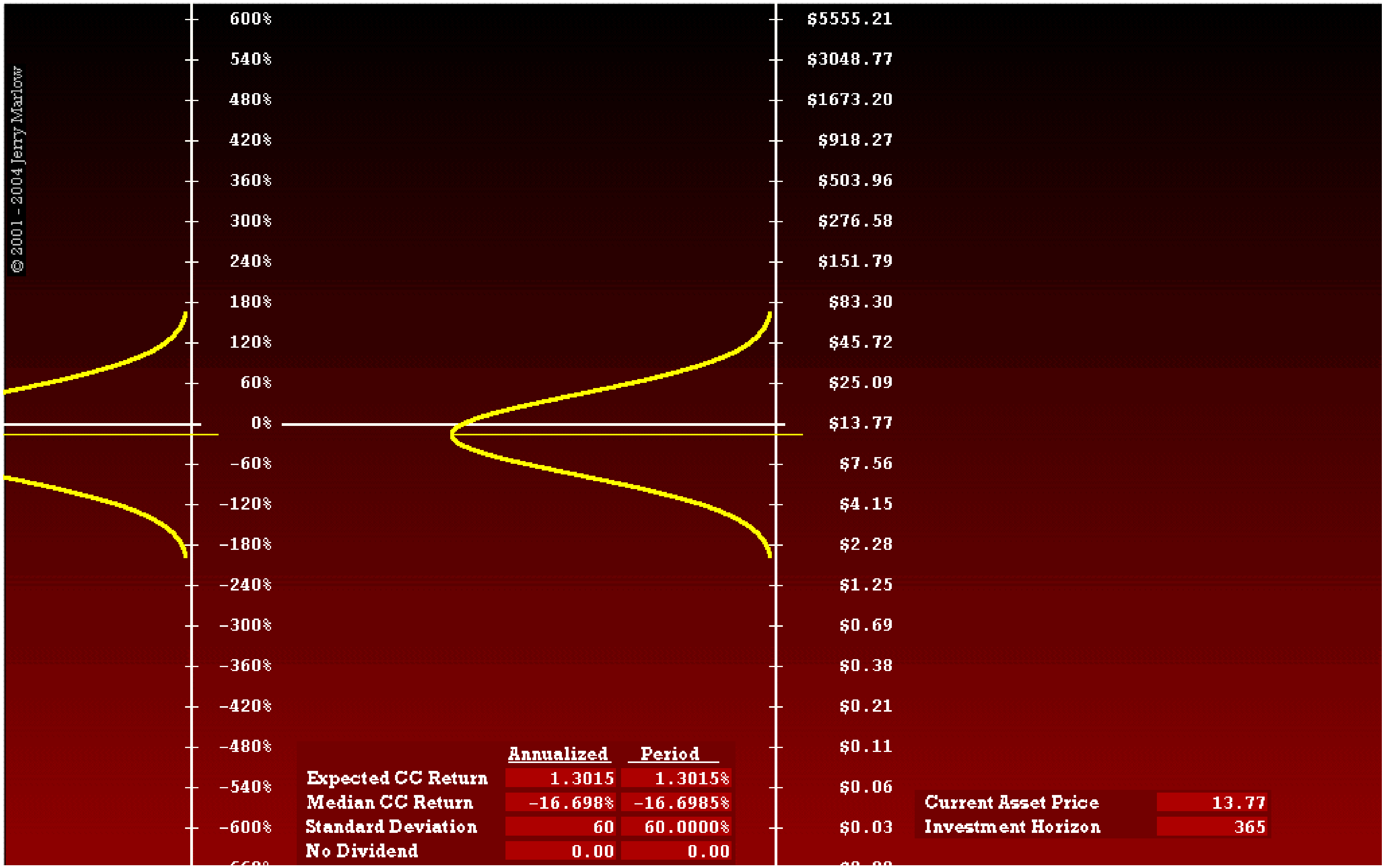
Accordingly more stock-price outcomes— more little squares— are above the strike price. They are farther above the strike price. There are more positive payoffs. The payoffs are larger. For the same $15.00 strike price, the cumulative probability-weighted present value of the potential payoffs is $7.92.



A less spread out bell-shaped curve centered at the same height gives a lower option value.

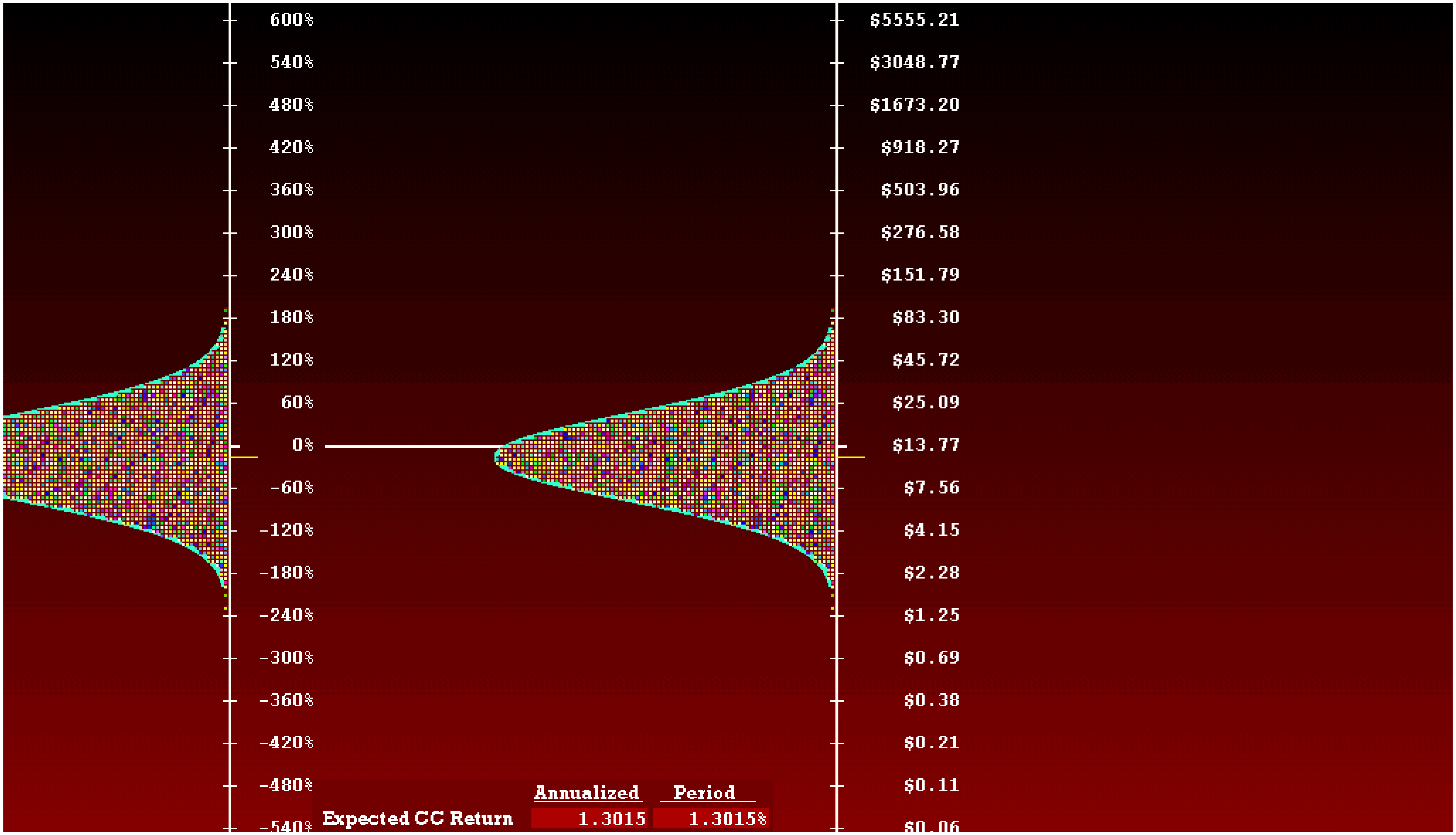
The bell-shaped curve shown here is centered at the same height on the stock-price axis but is less spread out. Fewer stock-price outcomes—fewer little squares— are above the strike price. Those that are above the strike price are not as far above.

For the same $15.00 strike price, the cumulative probability-weighted present value of the potential payoffs is $1.31.



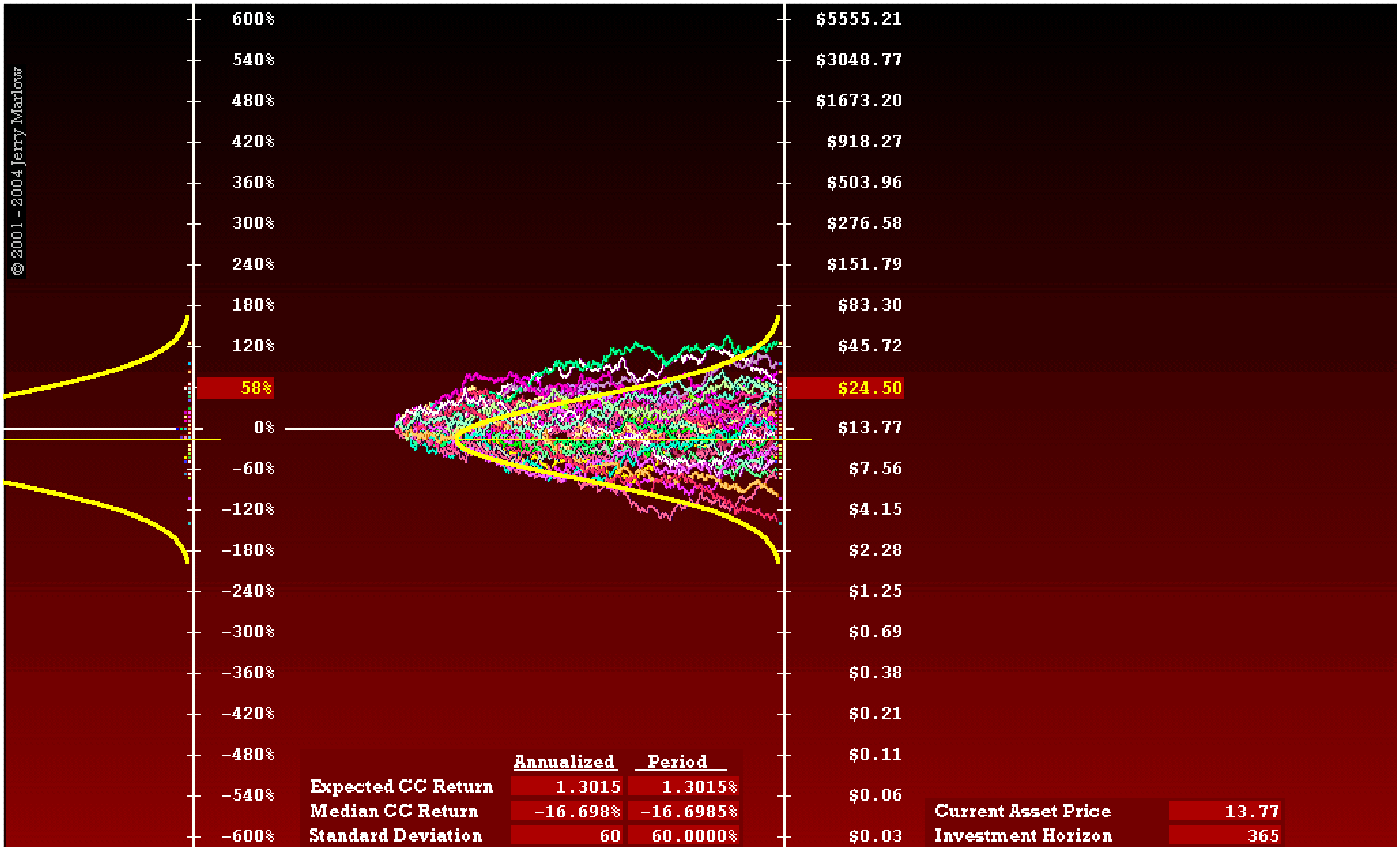
To draw a stock forecast, we need to know its expected return, standard deviation of expected volatility, current stock price, dividend forecast and the forecast's time horizon.

A stock's *return* forecast can be expressed in just two components: expected return and standard deviation of expected volatility. These are quoted in annual terms. To draw the forecast for a specific time horizon, we need to know its duration. To draw a price forecast, we also need to know the current stock price and the stock's dividend forecast.



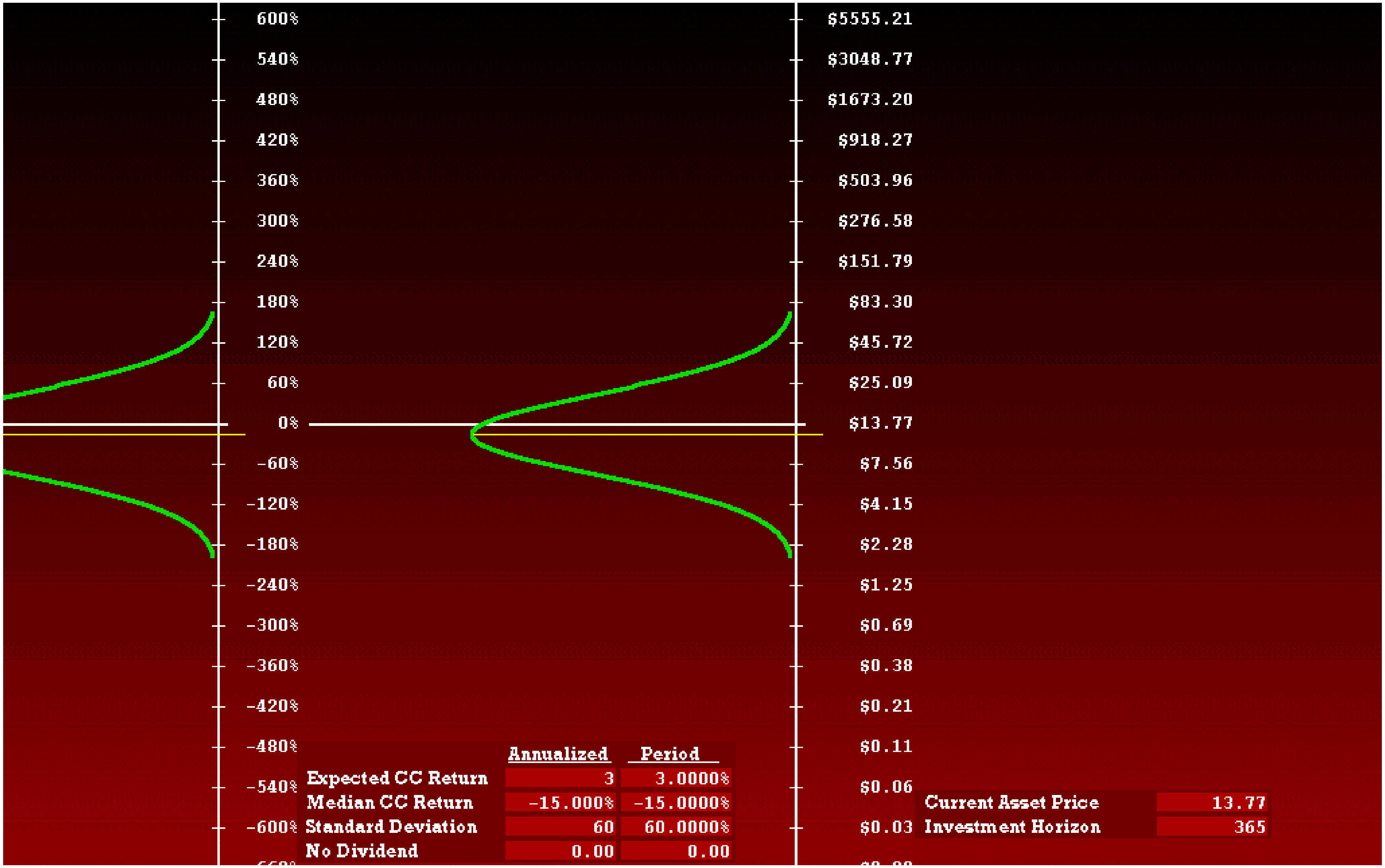
A stock's expected return is the average of all the returns in its probability distribution.

Expected return is the per-year *average* rate of return of all the little squares in the bell-shaped curve. In option valuation, we express expected return as a continuously compounded rate of return. In our initial valuation forecast, we used an expected return of 1.3015%, which happened to be equal to the then current one-year risk-free rate of return.



Standard deviation of expected volatility tells us how spread out the bell-shaped curve is.

The standard deviation of expected volatility quantifies how much we can expect the stock price to jump around. The greater the expected volatility, the greater the uncertainty about where the price will end up. The greater the uncertainty, the more spread out the bell-shaped probability distribution. In our initial valuation forecast, the standard deviation of expected volatility was 60%, as it is here.



In the bell-shaped curve, the median return is less than the average or expected return.

Here we draw a stock forecast with an expected return of 3% and a standard deviation of expected volatility of 60%. Yet, we see that the middle of the bell-shaped curve is down at -15%. More little squares are below than above the current stock price of $13.77.

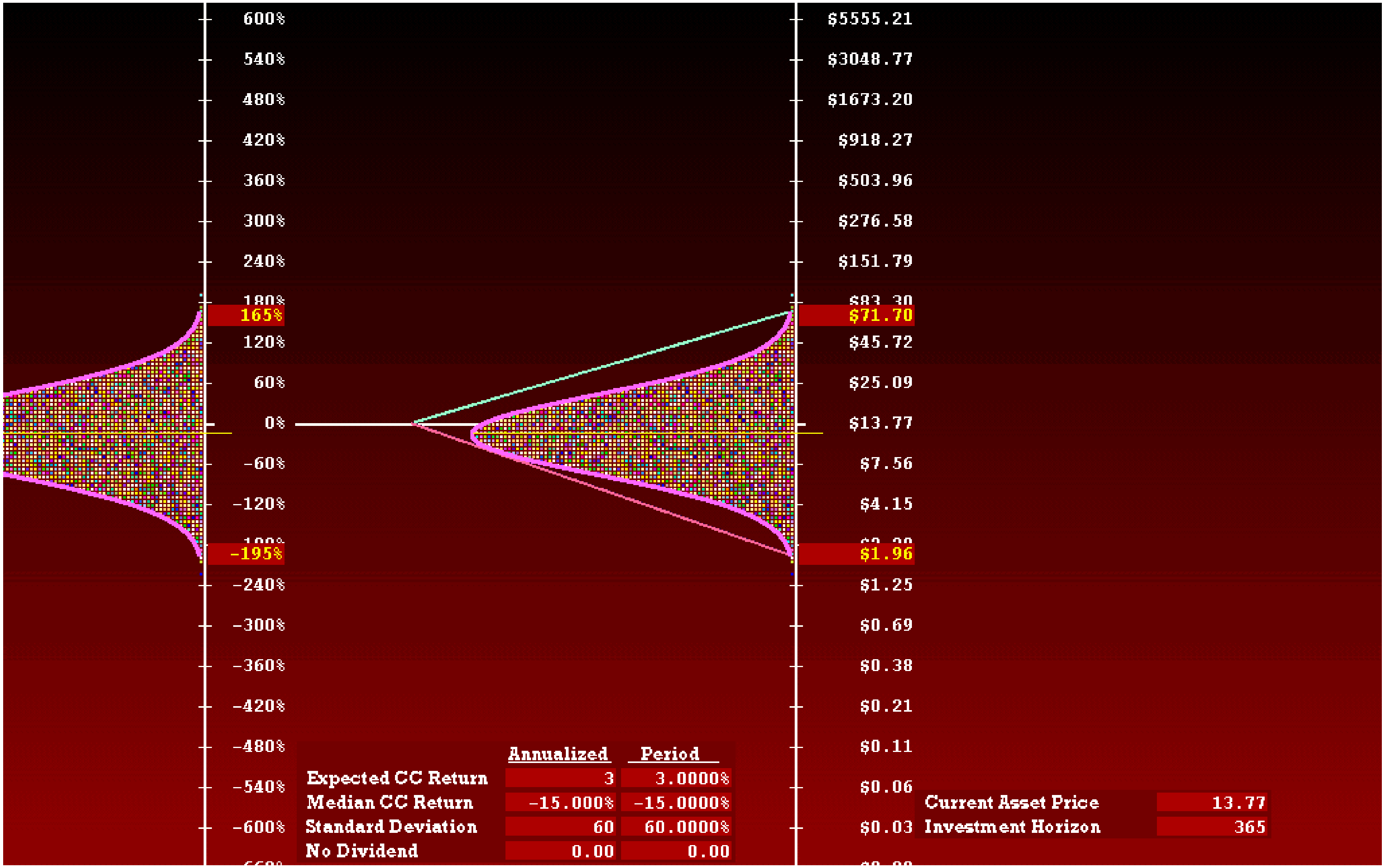
When working with continuously compounded rates of return, the average or expected return is not the same as the middle of the bell-shaped curve. To get a feel for why, imagine a portfolio of two stocks. The current price of each stock is $100. The value of the portfolio is $200. Over the course of a year, one stock has a continuously compounded return of 69%; the other has a continuously compounded return of –69%, that is, a loss of 69%.

The stock with a continuously compounded return of 69% doubles in value to $200. The stock with a continuously compounded return of -69% loses half its value and goes to $50. Hence, with one $100 stock going up by 69% and the other going down by 69%, the value of the portfolio goes from $200 to $250— for an average continuously compounded return of 22.31%.

When working with bell-shaped curves and continuously compounded rates of return, expected return and the standard deviation of volatility combine to tell us where the middle of the bell-shaped curve will fall. The middle or median return is equal to the expected return minus half the standard deviation squared. In our example:

$$
\begin{aligned}
\text{Median Return} &= ER - .5(SD^2) \\
&= .03 - .5(.60^2) \\
&= .03 - .5(.36) \\
&= .03 - .18 \\
&= -.15 \\
&= -15\%
\end{aligned}
$$

If a forecast's median return is negative, as it is here, the forecast says that the probability that the stock price will go down is greater than the probability that it will go up. More little squares in the bell-shaped curve fall below the current price level than above it.



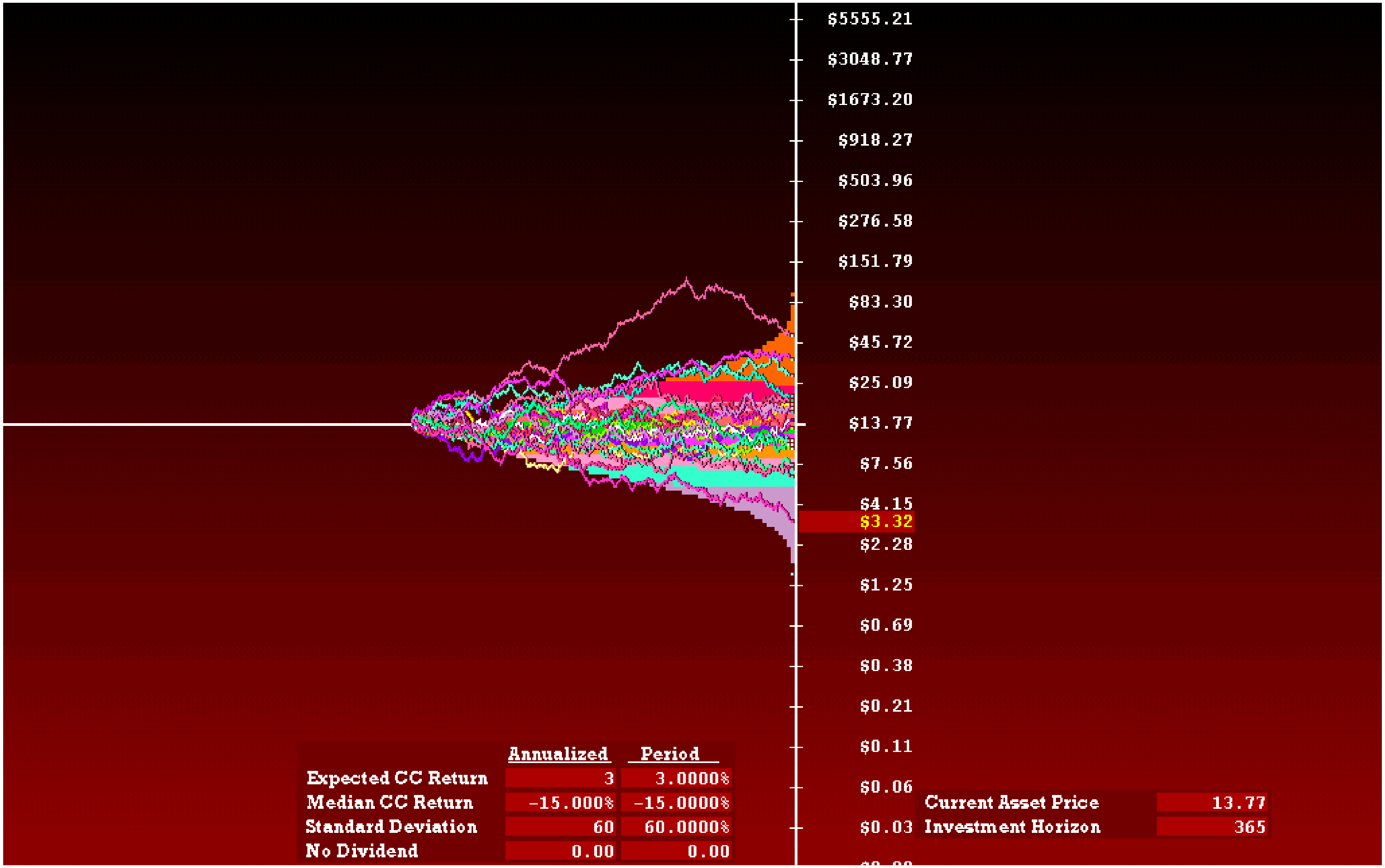
We can expect 99.87% of outcomes to fall between three standard deviations below the median return and three standard deviations above.

The forecast median return and standard deviation tell us the range within which we can be confident stock prices will be at the end of the forecast horizon. Under our statistical methodology, we can be 99.87% confident that the price will fall somewhere between three standard deviations below the median return and three standard deviations above.

In our example, we see that almost all the forecast returns fall above –195%, which is the median return minus three standard deviations:
-15% - (3 x 60%) = –195%.

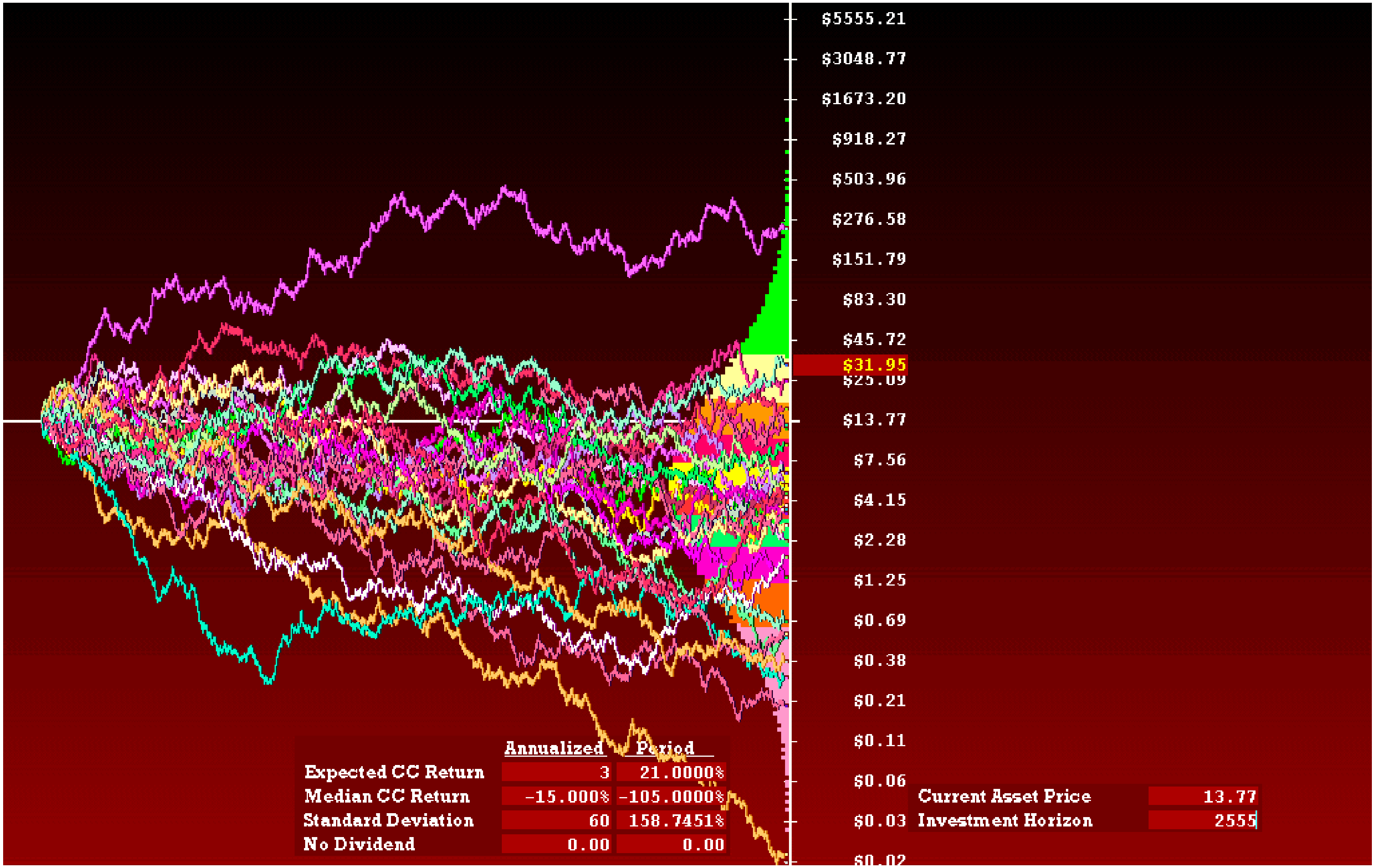
We see that almost all returns fall below 165%, which is the median return plus three standard deviations:
-15% + (3 x 60%) = 165%.

Looking across to the price axis, we see that the return range of -195% to 165% corresponds to a price range of $1.96 to $71.70. Hence, if we accept this forecast, we can be 99.87% confident that, one year from now, the price of this stock will be somewhere between $1.96 and $71.70.



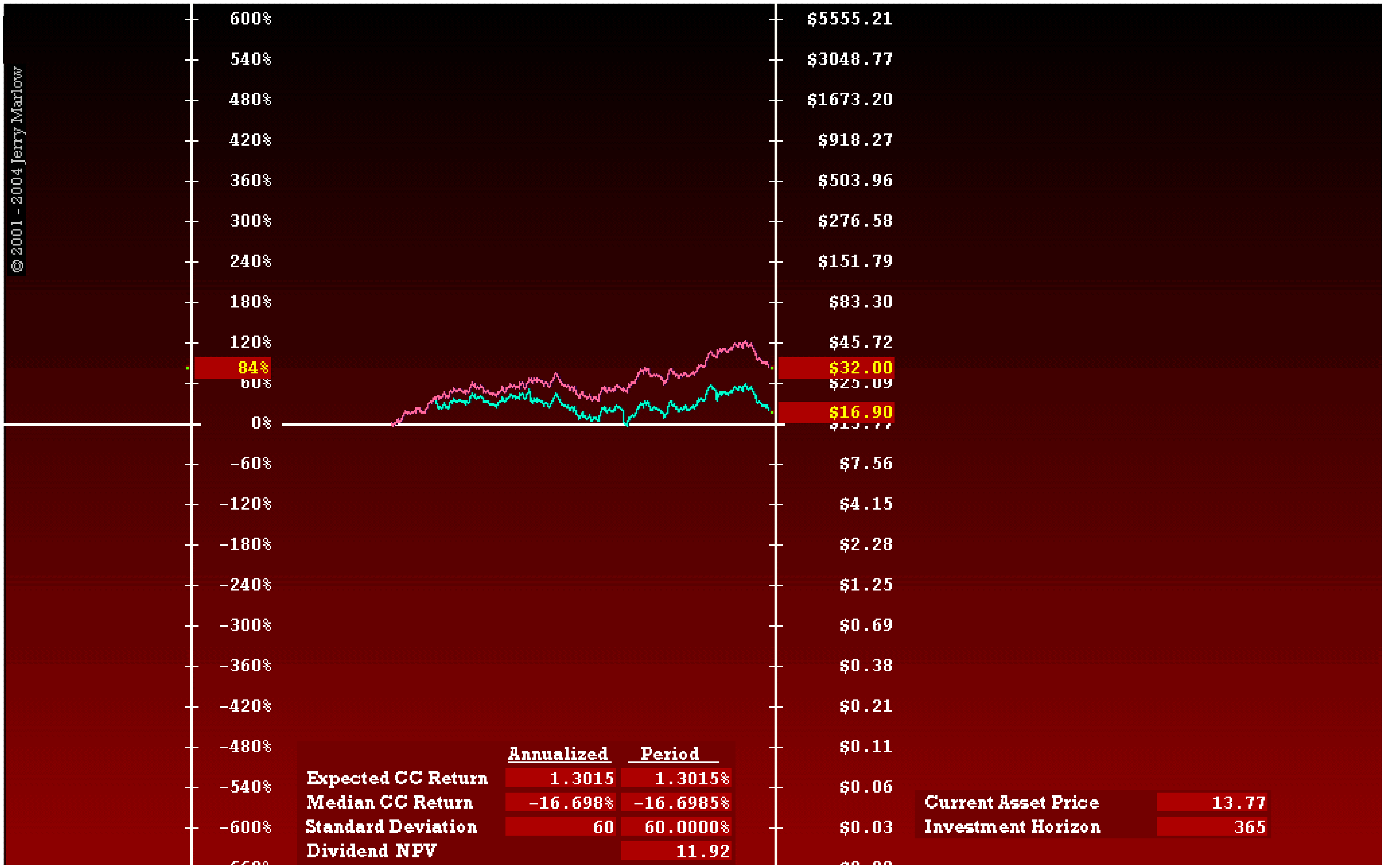
The longer the forecast horizon, the more spread out the bell-shaped curve.

As we do again here, we've been looking at price-path simulations and bell-shaped curves for one-year investment horizons. For a given standard deviation of expected volatility over a given investment horizon, a stock price has only so much time to jump around. It can go only so high or so low. The longer the investment horizon, the more time the stock price has to jump around, the higher or lower it can go, and the more spread out the bell-shaped curve.



The longer an option's time to expiration, the greater its value.

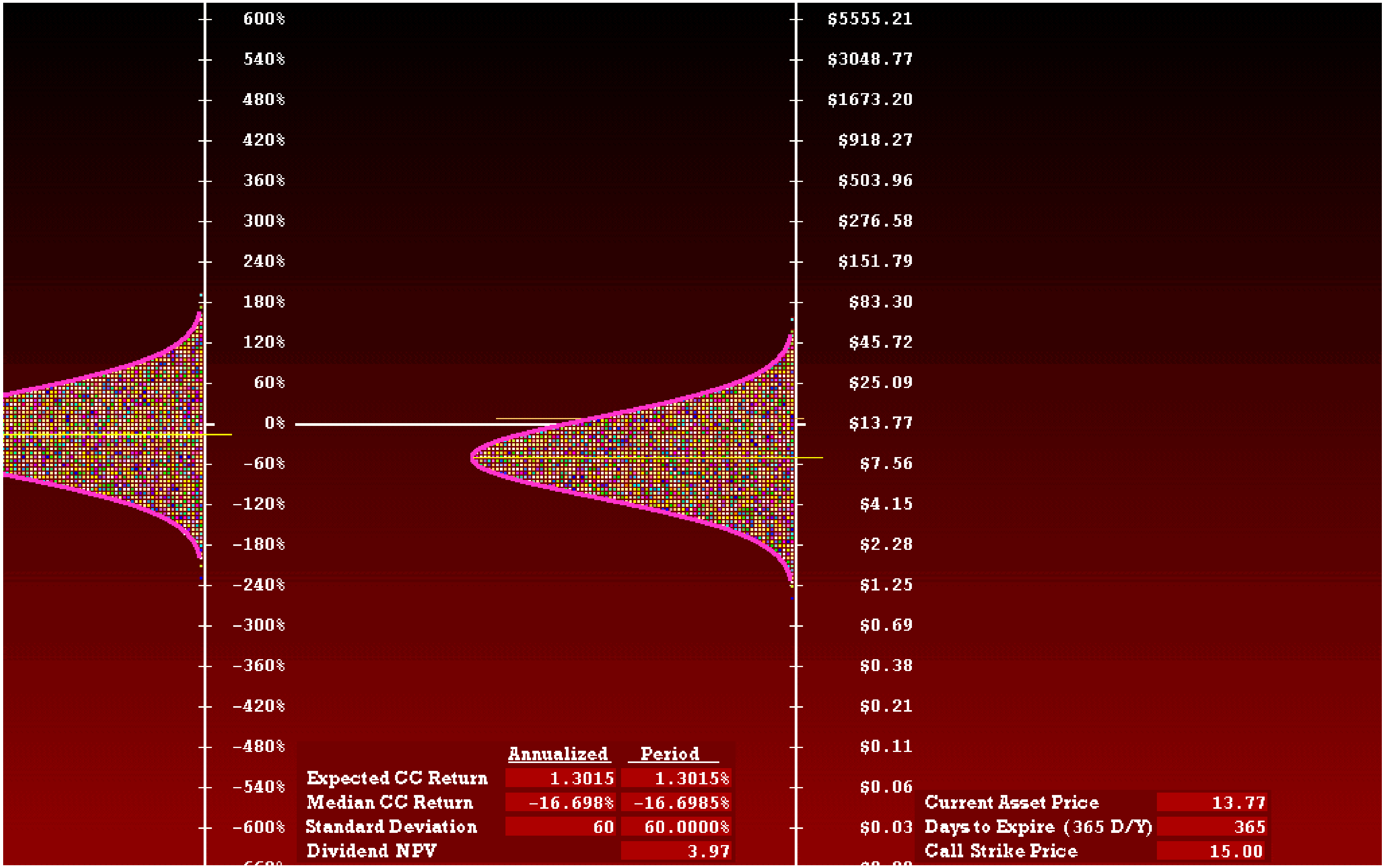
Here we look at the same forecast of expected return and volatility over a seven-year investment horizon. The stock price has more time to stray from where it is at time zero. In the bell-shaped curve, some of the potential price outcomes will be much farther above the strike price than they are in the same forecast over a one-year period. These more extreme potential payoffs give an option a value greater than that of one with a shorter time to expiration.



If a stock pays dividends, its price path will be lower than it otherwise would have been.

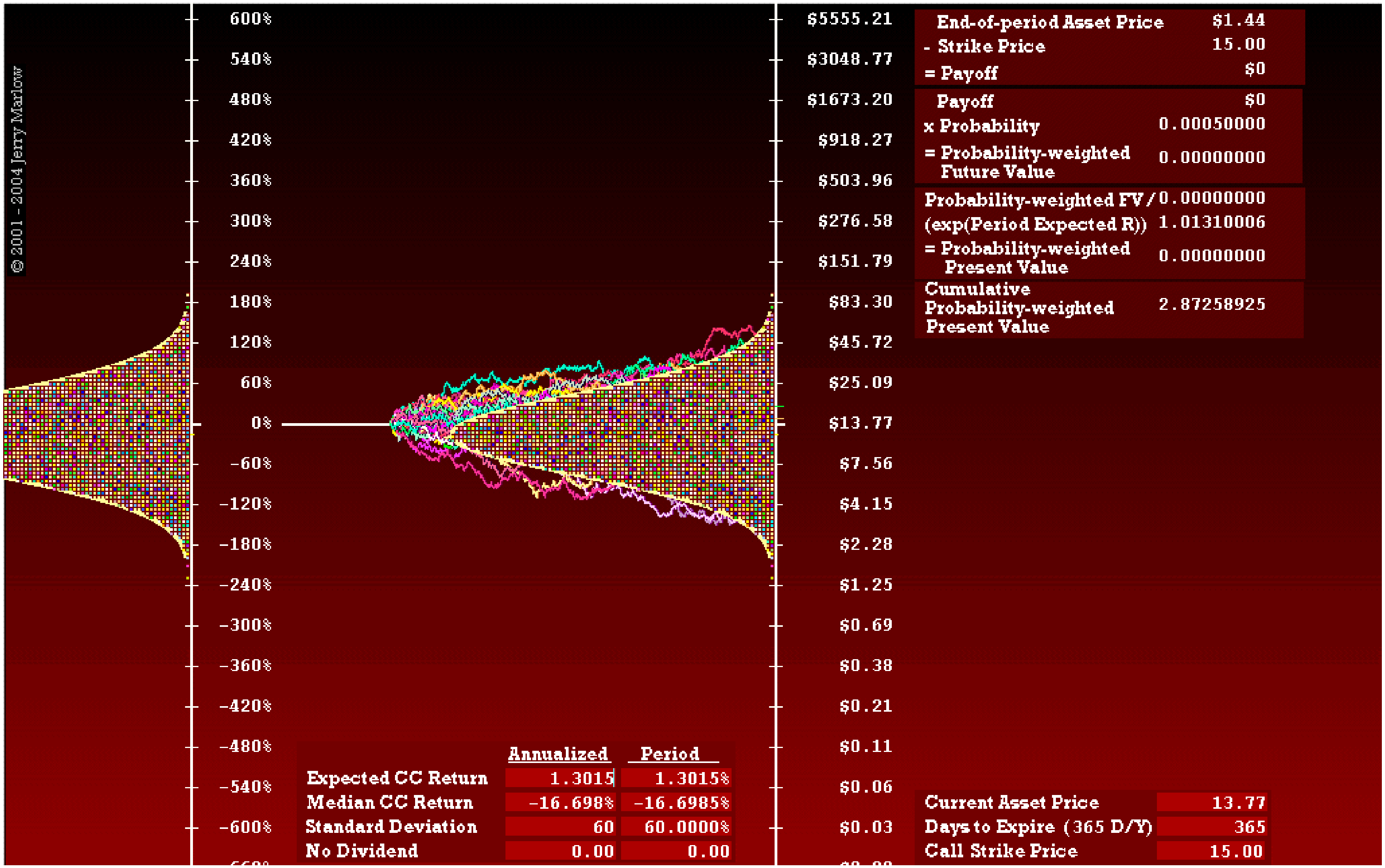
The forecast from which we simulate price paths and draw probability distributions is a return forecast. Stock returns include both price appreciation of the stock and dividend payments. Accordingly, any dividend payments come out of the price of the stock.

For sake of illustration, we simulate here a price path with and without four overly large quarterly dividend payments of $3.00 each.



If a stock pays dividends, the price bell-shaped curve will sit lower than the return curve.

Because dividend payments drop a stock's future price below what it otherwise would be, they shift the future-price bell-shaped curve below the return bell-shaped curve. All price outcomes in the forecast— all little squares— are lower than they otherwise would be. Option payoffs are based on stock price, not stock return. Hence, dividend payments lower call-option payoffs and values below what they otherwise would be.



This methodology allows us to value options fairly in the face of uncertainty.

Looking back on and rearranging what we have done, we see that, given a stock forecast of expected return and expected volatility and a discount rate, we can calculate an option's probability-weighted present value. That value takes into account all the prices that the stock *might* have at the end of the investment horizon. With this methodology, we can value options fairly in the face of uncertainty.

How does an option's probability-weighted present value compare to its Black-Scholes value?

Expert witnesses often recommend that, to value employee stock options, courts rely upon the Black-Scholes methodology. How does an option's probability-weighted present value compare with its Black-Scholes value?

To calculate an option's probability-weighted present value, we perform thousands of calculations, but the mathematics of each calculation is relatively easy to understand. By contrast, the Black-Scholes formula requires very few computations, but the mathematics of the formula can be difficult for the non-mathematician to comprehend.

To work with and have faith in Black-Scholes values, however, courts, attorneys and their clients need not understand the Black-Scholes mathematics because the Black-Scholes formula gives the same option value as does calculating the option's probability-weighted present value. To bring the two values into alignment, all we have to do is set the expected return of the stock forecast equal to the risk-free rate of return and divide the bell-shaped curve into a sufficiently large number of little squares.

The Black-Scholes-Merton formula for call options

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

S_0 =Stock price at time zero

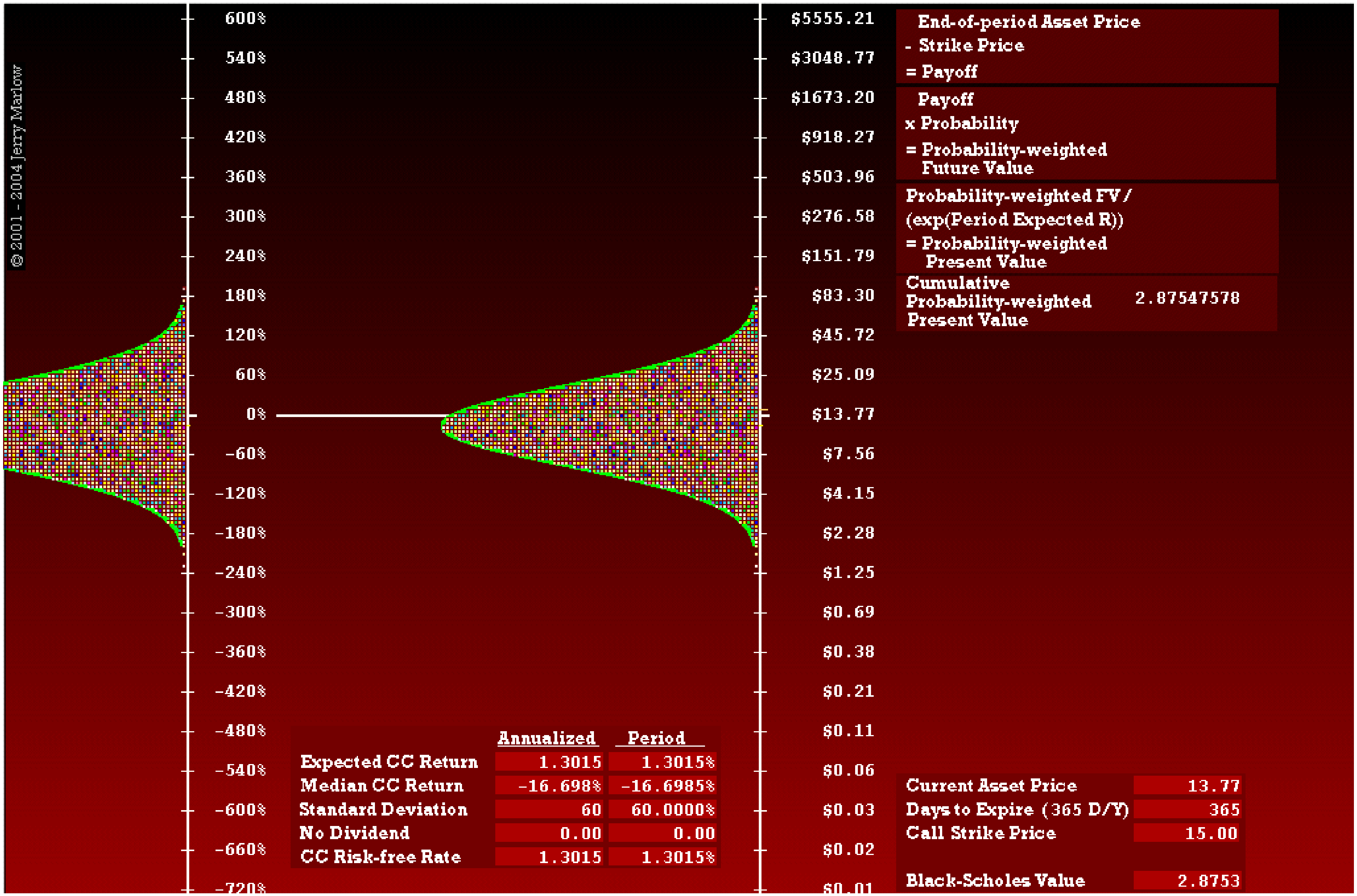
r = Continuously compounded risk-free rate

T = Option's time to expiration

N(x) = The cumulative normal distribution function

K = The option's strike price

σ = Volatility of the relative price change of the underlying stock price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$2.28
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$0.69
$0.38
$0.21
$0.11
$0.06
$0.03
$0.02
$0.01
End-of-period Asset Price
- Strike Price
= Payoff
Payoff
x Probability
= Probability-weighted Future Value
Probability-weighted FV / (exp(Period Expected R))
= Probability-weighted Present Value
Cumulative Probability-weighted Present Value 2.87547578
Annualized Period
Expected CC Return 1.3015 1.3015%
Median CC Return -16.698% -16.6985%
Standard Deviation 60 60.0000%
No Dividend 0.00 0.00
CC Risk-free Rate 1.3015 1.3015%
Current Asset Price 13.77
Days to Expire (365 D/Y) 365
Call Strike Price 15.00
Black-Scholes Value 2.8753

To make an option's probability-weighted present value equal its Black-Scholes value, set the stock forecast's expected return equal to the risk-free rate.

In our valuation example, we set the stock forecast's expected return equal to the risk-free rate. When we divided the bell-shaped curve into 100,000 little squares, we got a cumulative probability-weighted present value of all the option payoffs of $2.875. When we calculate the Black-Scholes value of the same option, we get a Black-Scholes value of $2.875. The two values are the same.

Even though the two values are the same, the Black-Scholes methodology provides a different way of thinking about an option's value.

An option's Black-Scholes value is the cost to a trader of setting up a delta hedge against the sale of the option

When an options trader sells options, ordinarily he or she does not wish to be exposed to a gain or loss depending on the option's payoff. Instead, the trader sets up a hedge on the sale and earns his or her profit on the difference between the bid and ask prices he or she offers.

A hedge is a transaction or series of transactions that eliminates a person's risk of loss due to changes in market prices. In eliminating the risk of loss, a hedge also eliminates the opportunity for gain due to changes in market prices.

For example, a farmer may calculate that she can bring 10,000 bushels of corn to market six months hence at a cost of $1.80 per bushel. The current market price of corn is $2.00 per bushel, but no one knows what the market price will be six months hence. If the farmer does not hedge her position and, at the time she brings her corn to market, the price is below $1.80 per bushel, she will suffer a loss.

To avoid the possibility of a loss, she can enter into a forward contract today to sell corn six months hence for a set price. So long as that price is above her cost of bringing the corn to market, she will earn a profit. She will not be affected negatively or positively by where the market price is six months hence.

The Black-Scholes methodology provides a way— called delta hedging— for a trader to hedge option sales in the face of uncertainty about where the price of the underlying stock is going to go.

Setting up a delta hedge is more complicated than hedging corn. Also, as the market price of the stock changes, the trader must continually rebalance his or her hedge. Though a trader must continuously rebalance the hedge, under the Black-Scholes assumptions, rebalancing is cost free. Hence, the cost of the hedge is the cost of setting it up. The cost of setting up the hedge is the fair value of the option.

When a trader sets up a delta hedge, his assets are delta shares of stock

To set up a delta hedge, a trader buys delta shares of stock. An option's delta is the ratio of a change in the option value to a change in the market price of the underlying stock. If, for example, when the market price of the stock goes up by one cent, the value of the option goes up by 0.571068 cent, then delta equals 0.571068.

An option's delta remains the same only over a small price range for the stock. As the price of the stock changes, delta changes. Also, as the number of days to expiration changes, an option's delta changes. A call option's delta always lies somewhere between 1.0 and 0.0

When the market price of a stock is significantly above a call option's strike price, we say that the option is deep in the money. If an option is deep in the money and little time is left before it expires, delta will be close to 1.0. That is, a one-cent change in the price of the stock will cause close to a one-cent change in the value of the call.

When the market price of a stock is significantly below a call option's strike price, we say the option is deep out of the money. If an option is deep out of the money and little time is left before it expires, delta will be close to 0. A one-cent change in the price of the stock will cause almost zero change in the value of the option.

When a trader sets up a delta hedge, his liabilities are the money he has borrowed to buy the shares of stock and the value of the options.

To set up the hedge, a trader buys delta shares of stock. He also borrows an amount of money equal to the present value of the probability that the option will finish in the money times the option's strike price times the number of options.

The probability that an option will finish in the money remains the same only over a small price range for the stock. As the price of the stock changes, the probability changes. Also as the number of days to expiration changes, the probability changes. The probability always lies somewhere between 0.0 and 1.0.

If an option is deep in the money and little time is left before the option expires, the probability that the option will finish in the money will be close to 1.0. If an option is deep out of the money and little time is left before the option expires, the probability that it will finish in the money will be close to 0.0.

The amount of money the trader borrows is insufficient to pay for delta shares of stock. The shortfall is the amount that the trader charges for the option. Hence, the shortfall is the dollar cost of setting up the hedge. The cost of setting up the hedge is the option's Black-Scholes value.

When a trader sets up a delta hedge, his assets and liabilities are in balance. Delta hedging keeps them in balance with no additional infusion of cash from the trader.

With the hedge set up in this way, the trader's assets and liabilities are in balance. His assets are delta shares of stock. His liabilities are the money he has borrowed plus the value of the options.

As the market price of the stock changes and as the time to expiration changes, delta changes and the probability that the option will finish in the money changes. As delta and the probability change, the hedge requirements change. To remain hedged, the trader must rebalance his assets and liabilities.

If the market price of the stock goes up, then delta goes up and the probability that the option will finish in the money goes up. The trader borrows more money and buys more shares of stock.

If the market price of the stock goes down, then delta goes down and the probability that the option will finish in the money goes down. The trader sells some of his shares of stock and pays down some of his loan.

If the market price of the stock remains the same and changes in the time to expiration cause delta and the probability that the stock will finish in the money to change, then the trader adjusts his hedge accordingly.

Rebalancing the hedge is cost free to the trader. His assets and liabilities change but they remain in balance.

When an option expires, the trader's assets and liabilities go to zero.

If, at expiration, the market price of the stock is above the strike price, then delta equals 1.0. The trader will own one share of stock for every option he has sold. The probability that the market price will be above the strike price will be 1.0. The amount of the loan will equal the strike price times the number of options.

If the option finishes in the money, the investor will exercise it. She pays the trader the strike price times the number of options. The trader delivers the shares to the investor. The trader uses the proceeds of the sale to pay off his loan plus interest— which at this point will also equal the strike price times the number of options. His assets go to zero. His liabilities go to zero.

If, on the other hand, at expiration, the market price of the stock is below the strike price, then delta equals zero. The trader owns no stock. The probability that the market price will finish above the strike price is zero. The value of the loan is zero. The investor does not exercise her options. Here too, the trader ends up with no assets and no liabilities.

Delta Hedging— An Example

We illustrate delta hedging with an example.

At the end of a trading day, an investor buys 10,000 call options with these characteristics:

Strike price:	\$14.00
Days until expiration:	5

The underlying stock has these characteristics:

Current market price of stock:	\$13.77
Standard deviation of volatility	60%

The underlying stock pays no dividends. The continuously compounded risk-free rate is 1.3015%.

On Expiration Day Minus 5

To set up the delta hedge, using the Black-Scholes formula, the options trader calculates the option's delta and the probability that the option will finish in the money. He then makes the following calculations:

	Option Delta	0.421431
X	Current Stock Price	$13.770000
=	Cost of Delta Share	$5.803101
	Probability Finish in Money	0.394173
X	Strike Price	$14.000000
=	Bond Value at Expiration	$-5.518419
	Bond Value at Expiration	$-5.518419
/	exp(Period Risk-free Rate)	1.000178
=	Bond Value Today	$-5.517435
	Cost of Delta Share	$5.803101
+	Bond Value Today	$-5.517435
=	Black Scholes Value	$0.285666

The number of shares the trader buys is given by: Delta x 10,000 = 4,214.31.

(To make the numbers come out right, we allow for the purchases and sale of fractional shares.)

With the market price of the stock equal to $13.77, the amount of money the trader pays for the shares is given by:
4,214.31 x $13.77 = $58,031

The amount of money the trader borrows is given by:
Bond Value Today x 10,000 = $55,174

The amount of money he receives for the options is given by:
Black-Scholes value x 10,000 = $2,857

The trader's assets are the shares he owns. His liabilities are the bond value he owes plus the market value of the options he has sold:
$55,174 + $2,857 = $58,031

His assets and liabilities are in balance.

In our example, instead of rebalancing the hedge whenever delta changes, we balance at the end of each day.

On Expiration Day Minus 4

At the end of the next day, the market price of the stock has gone up to $14.20. The trader calculates the option's new delta and probability that the option will finish in the money. He then makes these calculations:

	Option Delta	0.602378
X	Current Stock Price	$14.200000
=	Cost of Delta Share	$8.553768
	Probability Finish in Money	0.577967
X	Strike Price	$14.000000
=	Bond Value at Expiration	$-8.091543
	Bond Value at Expiration	$-8.091543
/	exp(Period Risk-free Rate)	1.000143
=	Bond Value Today	$-8.090389
	Cost of Delta Share	$8.553768
+	Bond Value Today	$-8.090389
=	Black Scholes Value	$0.463380

To maintain his delta hedge, the trader needs to modify his position so that it will be consistent with the new delta and probability that the option will finish in the money. He now needs to own 0.602378 x 10,000 = 6,023.78 shares. Currently he owns 4,214.31. So he buys 6,023.78 - 4,214.31 = 1,809.47 more. At the new market price, they cost him 1,809.47 x $14.20 = $25,694. The 6,023.78 shares he now owns, his assets, have a market value of 6,023.78 x $14.20 = $85,538.

The trader needs for the value of the bond to be $8.090389 x 10,000 = $80,904. Currently he owes the $55,174 he borrowed plus $2 in one-day interest. So he borrows $80,904 - $55,174 - $2 = $25,728 more. His bond balance is now $80,904.

The 10,000 options the trader sold now have a market value of 0.4634 x 10,000 = $4,634.

The trader's liabilities now equal the bond balance plus the market value of the options: $80,904 + $4,634 = $85,538.

The trader's assets and liabilities are in balance.

On Expiration Day Minus 3

At the end of the next day, the market price of the stock has gone up to $14.95. The trader calculates the option's new delta and probability that the option will finish in the money. He then makes these calculations:

	Option Delta	0.891795
X	Current Stock Price	$14.950000
=	Cost of Delta Share	$13.332340
	Probability Finish in Money	0.881345
X	Strike Price	$14.000000
=	Bond Value at Expiration	$-12.338828
	Bond Value at Expiration	$-12.338828
/	exp(Period Risk-free Rate)	1.000107
=	Bond Value Today	$-12.337508
	Cost of Delta Share	$13.332340
+	Bond Value Today	$-12.337508
=	Black Scholes Value	$0.994832

To maintain his delta hedge, the trader again needs to modify his position so that it will be consistent with the new delta and probability that the option will finish in the money. He now needs to own 0.891795 x 10,000 = 8,917.95 shares. Currently he owns 6,023.78. So he buys 8,917.95 - 6,023.78 = 2,894.17 more. At the new market price, they cost him 2,894.17 x $14.95 = $43,268. The 8,917.95 shares he now owns, his assets, have a market value of 8,917.95 x $14.95 = $133,323.

The trader needs for the value of the bond to be $12.337508 x 10,000 = $123,375. Currently he owes the $80,904 previous day's balance plus $3 in one-day interest. So he borrows $123,375 - $80,904 - $42,468 = $42,468 more. His bond balance is now $123,375.

The 10,000 options the trader sold now have a market value of $0.994832 x 10,000 = $9,948.

The trader's liabilities now equal the bond balance plus the market value of the options: $123,375 + $9,948 = $133,323.

The trader's assets and liabilities are in balance.

On Expiration Day Minus 2

At the end of the next day, the market price of the stock has gone down to $14.67. The trader calculates the option's new delta and probability that the option will finish in the money. He then makes these calculations:

	Option Delta	0.859114
X	Current Stock Price	$14.670000
=	Cost of Delta Share	$12.603202
	Probability Finish in Money	0.848948
X	Strike Price	$14.000000
=	Bond Value at Expiration	$-11.885276
	Bond Value at Expiration	$-11.885276
/	exp(Period Risk-free Rate)	1.000071
=	Bond Value Today	$-11.884428
	Cost of Delta Share	$12.603202
+	Bond Value Today	$-11.884428
=	Black Scholes Value	$0.718774

To maintain his delta hedge, the trader once again needs to modify his position so that it will be consistent with the new delta and probability that the option will finish in the money. To own delta shares, he now needs to own 0.859114 x 10,000 = 8,591.14 shares. Currently he owns more shares than this; he owns 8,917.95. So he sells
8,917.95 - 8,591.14 = 326.81. At the new market price, he receives 326.81 x $14.67 = $4794. The shares he now owns, his assets, have a market value of 8,917.95 x $14.67 = $126,032.

The trader needs for the value of the bond to be $11.884428 x 10,000 = $118,844, which is less than the $123,375 previous day's balance plus $4 in one-day interest he currently owes. So he repays $123,375 + $4 - $118,844 = $4,535. His bond balance is now $118,844.

The 10,000 options the trader sold now have a market value of $0.718774 x 10,000 = $7,188.

The trader's liabilities now equal the bond balance plus the market value of the options: $118,844 + $7,188 = $126,032.

The trader's assets and liabilities are in balance.

On Expiration Day Minus 1

At the end of the next day, the market price of the stock has gone up to $15.27. The trader calculates the option's new delta and probability that the option will finish in the money. He then makes these calculations:

	Option Delta	0.997297
X	Current Stock Price	$15.270000
=	Cost of Delta Share	$15.228720
	Probability Finish in Money	0.997023
X	Strike Price	$14.000000
=	Bond Value at Expiration	$-13.958327
	Bond Value at Expiration	$-13.958327
/	exp(Period Risk-free Rate)	1.000036
=	Bond Value Today	$-13.957829
	Cost of Delta Share	$15.228720
+	Bond Value Today	$-13.957829
=	Black Scholes Value	$1.270891

To maintain his delta hedge, the trader needs to modify his position so that it will be consistent with the new delta and probability that the option will finish in the money. He now needs to own 0.997297 x 10,000 = 9,972.97 shares. Currently he owns 8,591.14. So he buys 9,972.97 - 8,591.14 = 1,381.83 more. At the new market price, they cost him 1,381.83 x $15.27 = $21,101. The 9,972.97 shares he now owns, his assets, have a market value of 9,972.97 x $15.27 = $152,287.

The trader needs for the value of the bond to be $13.957829 x 10,000 = $139,578. Currently he owes the $118,844 previous day's balance plus $4 in one-day interest. So he borrows $139,578 - $118,844 - $4 = $20,730 more. His bond balance is now $139,578.

The 10,000 options the trader sold now have a market value of $1.270891 x 10,000 = $12,709.

The trader's liabilities now equal the bond balance plus the market value of the options: $139,578 + $12,709 = $152,287.

The trader's assets and liabilities are in balance.

On Expiration Day

At the end of the next day, the market price of the stock has gone down to $15.19. The option finishes in the money. Delta is 1.0. The probability that the option will finish in the money is 1.0.

	Option Delta	1.000000
X	Current Stock Price	$15.190000
=	Cost of Delta Share	$15.190000
	Probability Finish in Money	1.000000
X	Strike Price	$14.000000
=	Bond Value at Expiration	$-14.000000
	Bond Value at Expiration	$-14.000000
/	exp(Period Risk-free Rate)	1.000000
=	Bond Value Today	$-14.000000
	Cost of Delta Share	$15.190000
+	Bond Value Today	$-14.000000
=	Black Scholes Value	$1.190000

To prepare for settlement, the trader needs to modify his position so that it will be consistent with the new delta and probability that the option will finish in the money. He now needs to own 1.0 x 10,000 = 10,000 shares. Currently he owns 9,972.97. So he buys
10,000 - 9,972.97 = 27.03 more. At the new market price, they cost him 27.03 x $15.19 = $411. The 10,000 shares he now owns, his assets, have a market value of 10,000 x $15.19 = $151,900.

The trader needs for the value of the bond to be $14.00 x 10,000 = $140,000. Currently he owes the $139,578 previous day's balance plus $5 in one-day interest. So he borrows $140,000 - $139,578 - $5 = $417 more. His bond balance is now $140,000.

The 10,000 options the trader sold now have a market value of $1.19 x 10,000 = $11,900.

The trader's liabilities now equal the bond balance plus the market value of the options: $140,000 + $11,900 = $151,900.

The trader's assets and liabilities are in balance.

Settlement

At settlement, the investor exercises the options. The investor buys the 10,000 shares from the trader for $140,000. The trader uses the $140,000 to repay his $140,000 bond liability. The trader's asset balance of shares goes t zero. His option liability goes to zero. His bond liability goes to zero. His assets and liabilities are in balance.

The following table summarizes the hedging transactions.

Delta-Hedging Example Summarized

	Stock Price	Option Delta	Number of Shares	Market Value of Shares (Asset)	Prob Finish in Money	Bond Value (Liability)	Option Value	Value of 10,000 Options (Liability)	Total Liabilities
5-Day	$13.77	0.42143			0.39417		0.2857		
Actions:		Buy	4,214.31	$58,031	Borrow	$55,174	Sell	$2,857	
Balances:			4,214.31	$58,031		$55,174		$2,857	$58,031
4-Day	$14.20	0.60238			0.57797		0.4634		
Interest:						$2			
Actions:		Buy	1,809.47	$25,694	Borrow	$25,728			
Balances:			6,023.78	$85,538		$80,904		$4,634	$85,538
3-Day	$14.95	0.8918			0.88135		0.9948		
Interest:						$3			
Actions:		Buy	2,894.17	$43,268	Borrow	$42,468			
Balances:			8,917.95	$133,323		$123,375		$9,948	$133,323
2-Day	$14.67	0.85911			0.84895		0.7188		
Interest:						$4			
Actions:		Sell	(326.81)	($4,794)	Repay	($4,535)			
Balances:			8,591.14	$126,032		$118,844		$7,188	$126,032
1-Day	$15.27	0.9973			0.99702		1.2709		
Interest:						$4			
Actions:		Buy	1,381.83	$21,101	Borrow	$20,730			
Balances:			9,972.97	$152,287		$139,578		$12,709	$152,287
Expiration	$15.19	1.0000			1.0000		1.1900		
Interest:						$5			
Actions:		Buy	27.03	$411	Borrow	$417			
Balances:			10,000.00	$151,900		$140,000		$11,900	$151,900
Settlement		Sell	10,000.00	($151,900)	Repay	($140,000)	Redeem	($11,900)	
Balances:			0	$0		$0		$0	$0

The financial marketplace enforces options' arbitrage-free values

Because, under the Black-Scholes assumptions, a trader can sell an option for its Black-Scholes value and hedge the sale, any other price would create an opportunity for a trader to earn risk-free profits— in other words, an opportunity to perform arbitrage. If the market price for an option were higher than the cost of hedging its sale, traders would flood the market with options at the higher price and hedge the sales. They would earn risk-free profits. After a while, the oversupply of options would force the market price down. The opportunity to perform arbitrage would disappear. The supply of and demand for the options would reach equilibrium at the Black-Scholes or arbitrage-free value.

If the market price of an option were lower than the Black-Scholes value, arbitragers could buy the option and do a reverse hedge. Again, at the lower price, they could earn risk-free profits. Over time, their demand for options at the lower price would drive the option's market price up to the Black-Scholes value. Here too, the opportunity to perform arbitrage would disappear. Supply and demand would reach equilibrium at the Black-Scholes or arbitrage-free value.

The Black-Scholes methodology gives us arbitrage-free values for market-traded options

To the degree that the behavior of market participants adheres to Black-Scholes options pricing theory, traders' ability to hedge option sales and purchases drives option prices in the marketplace toward their Black-Scholes values. Hence, the Black-Scholes value of a market-traded option can be thought of as a value enforced in the marketplace by the elimination of arbitrage opportunities. Equilibrium prices in the marketplace are arbitrage-free values.

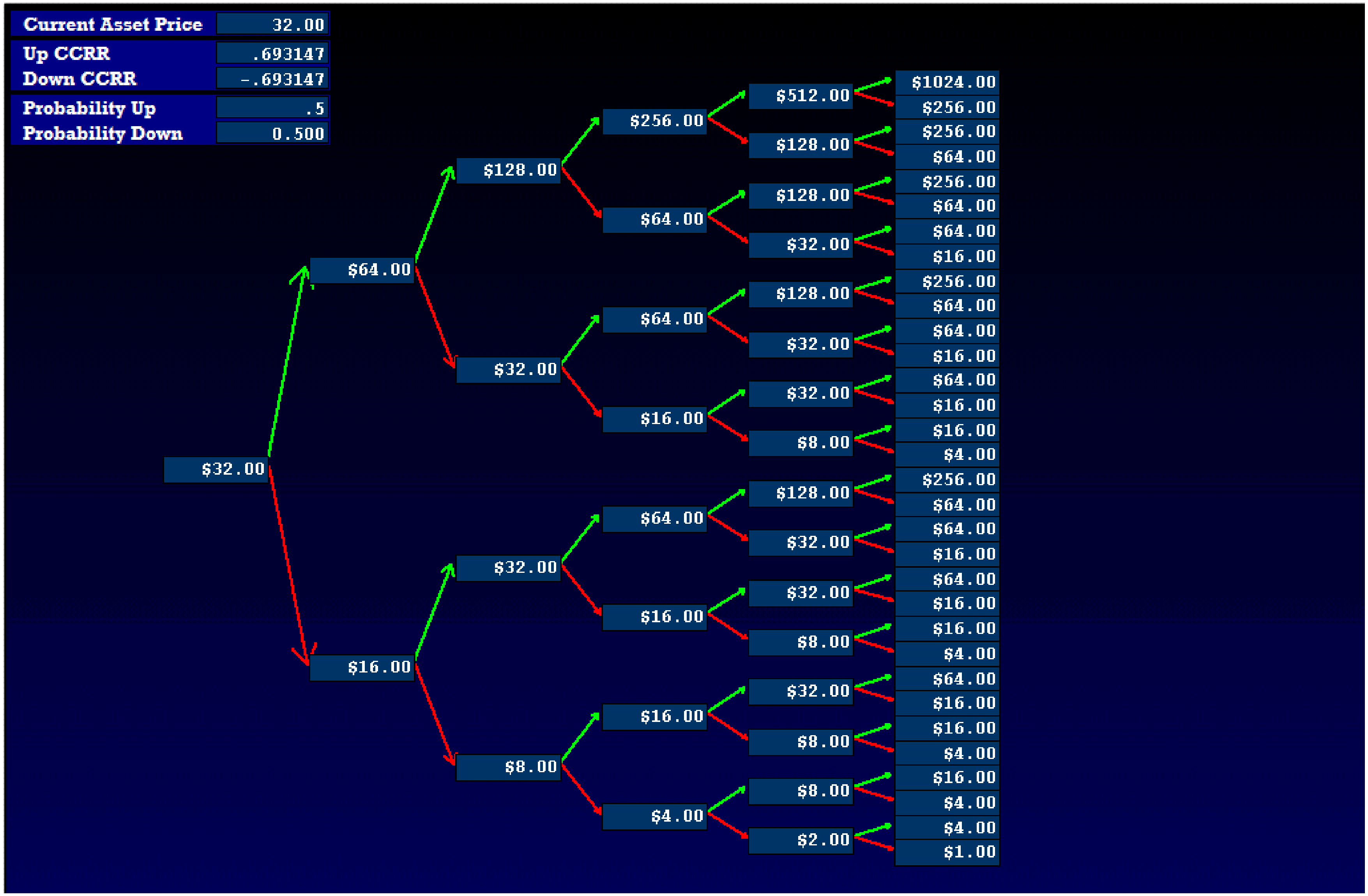
Risk-neutral valuation gives us arbitrage-free values for market-traded options

We saw that, when we set the stock forecast's expected return equal to the risk-free rate, then the option's probability-weighted present value is equal to its Black-Scholes value. The observation that one can, in this way, calculate arbitrage-free values for options gave rise to what are known as risk-neutral valuation techniques.

In risk-neutral valuation of options, the steps are the ones we followed initially when we calculated the option's probability-weighted present value:

1. Construct an arbitrage-free model of how stock prices might evolve over time.
2. Assign values to the model to account for the stock's volatility and for an expected return on the stock that is equal to the risk-free rate.
3. Model the stock's potential price paths and their probabilities.
4. For each price path, calculate the maximum expected payoff of the option and the payoff's probability.
5. Find the probability-weighted present value of all the potential payoffs.

You may have heard of or be familiar with one or more binomial option-pricing models. They are risk-neutral-valuation models. They follow these steps.



Modeling the evolution of a stock price on a binomial tree

If you're not familiar with binomial models, the easiest way to develop a feel for them is to imagine a stock forecast that says this: Each year over the next five years, the market price of the stock will either double or fall by half. Each year, the probability that the stock price will go up is 0.5; the probability that it will go down is 0.5.

If the price of the stock today is $32.00, then, over the coming year, either the price will rise to $64.00 or it will fall to $16.00. If, during the first year, the stock price has gone to $64.00, then, in the second year, the stock price either will go up to $128.00 or go down to $32.00, and so on. Thus, each year the stock price either doubles or falls by half.

When we look at the possible outcomes of this forecast over the full five years, we see that the price may go as high as $1,024.00; but that there's only one way to get there: The price has to go up at every node on its path. It has to double in value five times.

Likewise, we see that the price may go as low as $1.00; but there's only one way to get there: The price has to go down at every node on its path. It has to lose half its value five times.

We also see that there are thirty-two possible paths the stock price can follow. Hence, we can say that the probability that the stock price will go to $1,024 over five years is 1 in 32 or 1/32, or, expressed as decimal, 0.031. Likewise, the probability that the stock price will go all the way down to $1.00 is 1/32.

We also realize that, with five time steps and two possibilities at each node— up and down— the number of possible price paths, 32, is equal to 2^5.

When we look at the other possible outcomes at the end of the five years, we see that different paths can produce the same outcomes. Any path with four ups and one down produces a final price of $256.00. Any path with three ups and two downs produces a final price of $64. And so on. For any given number of ups and the rest downs, the order in which the ups and downs occur doesn't matter. The paths arrive at the same terminal price.

With five time steps, we see that, while there are thirty-two different possible price paths, there are only six different possible outcomes. When we count the number of outcomes at each price, we find this pattern:

Outcome	Number of Occurrences	Probability
$1,024.00	1	1/32
$256.00	5	5/32
$64.00	10	10/32
$16.00	10	10/32
$4.00	5	5/32
$1.00	1	1/32

From studying the binomial theorem in math class, you may remember that the coefficients of a binomial expansion follow the pattern known as Pascal's triangle.

	0		0		0		1		0		0		0	
0		0		0		1		1		0		0		0
	0		0		1		2		1		0		0	
0		0		1		3		3		1		0		0
	0		1		4		6		4		1		0	
0		1		5		10		10		5		1		0

In Pascal's triangle, in the top row, we start with a 1 in a field of zeroes. Then, on each subsequent row, numbers are equal to the sums of the closest two numbers above. After five steps, we get the pattern we found at the end of our five-step binomial tree:
1, 5, 10, 10, 5, 1.

Applied to binomial expansions, Pascal's triangle gives this pattern of coefficients:

$(a + b)^0 = 1$

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

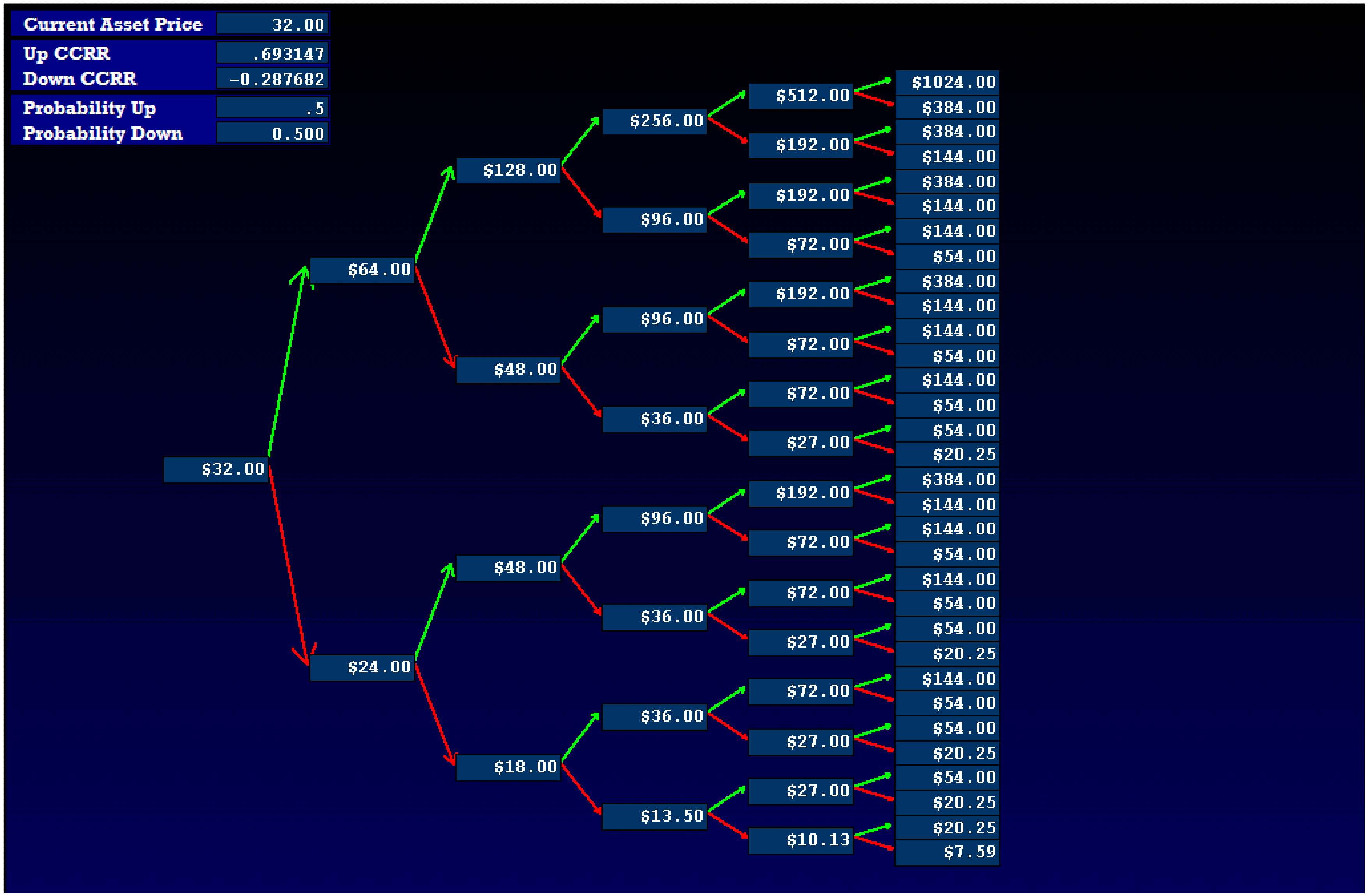
$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

To get the price to either double or lose half its value at each node, we set the up continuously compounded rate of return equal to 69.3147% and the down continuously compounded rate of return equal to -69.3147%.

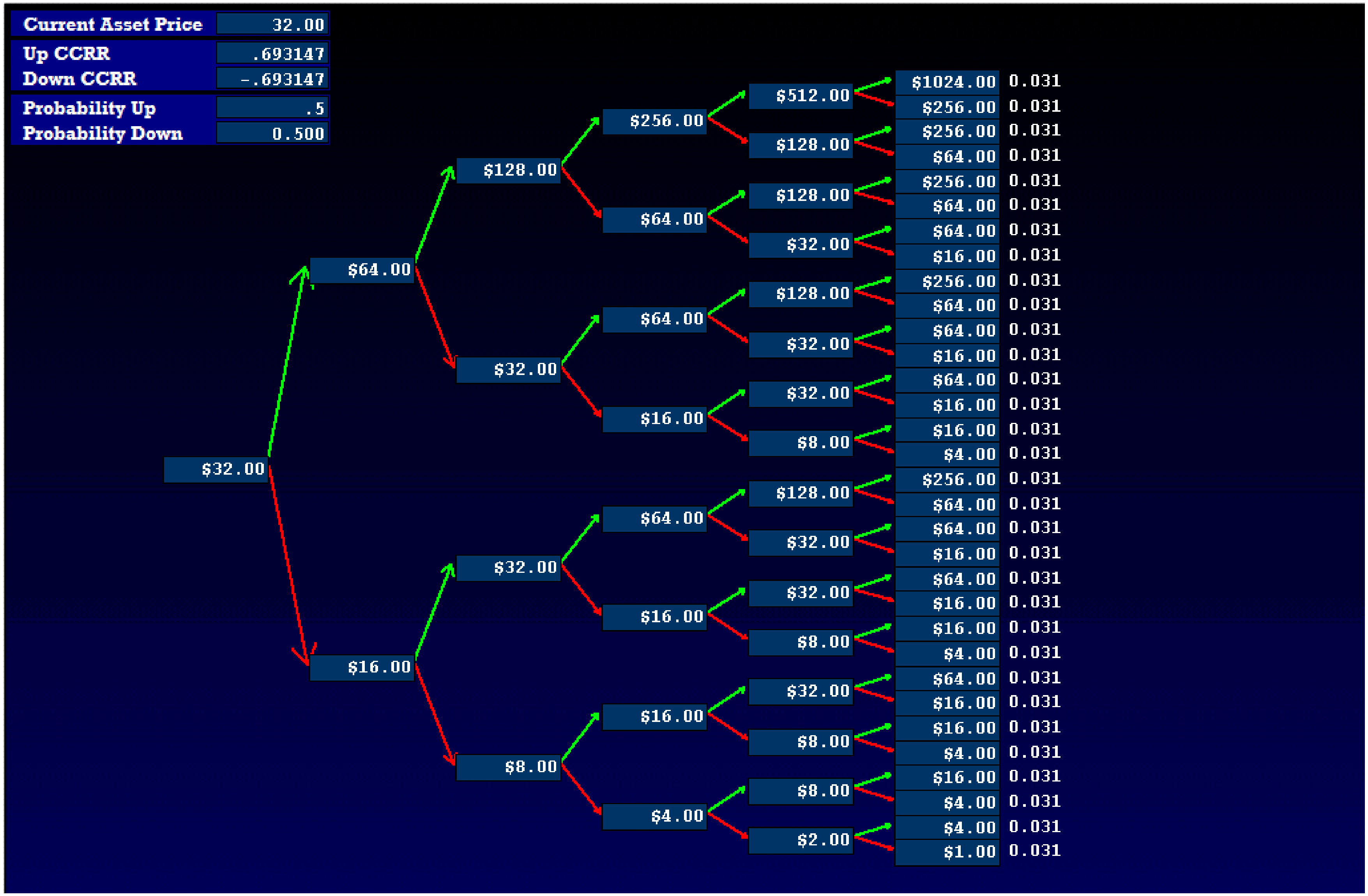
log(2.0) = 69.3147%
exp(69.3147%) = 2.0

log(0.5) = -69.3147%
exp(-69.3147%) = 0.5

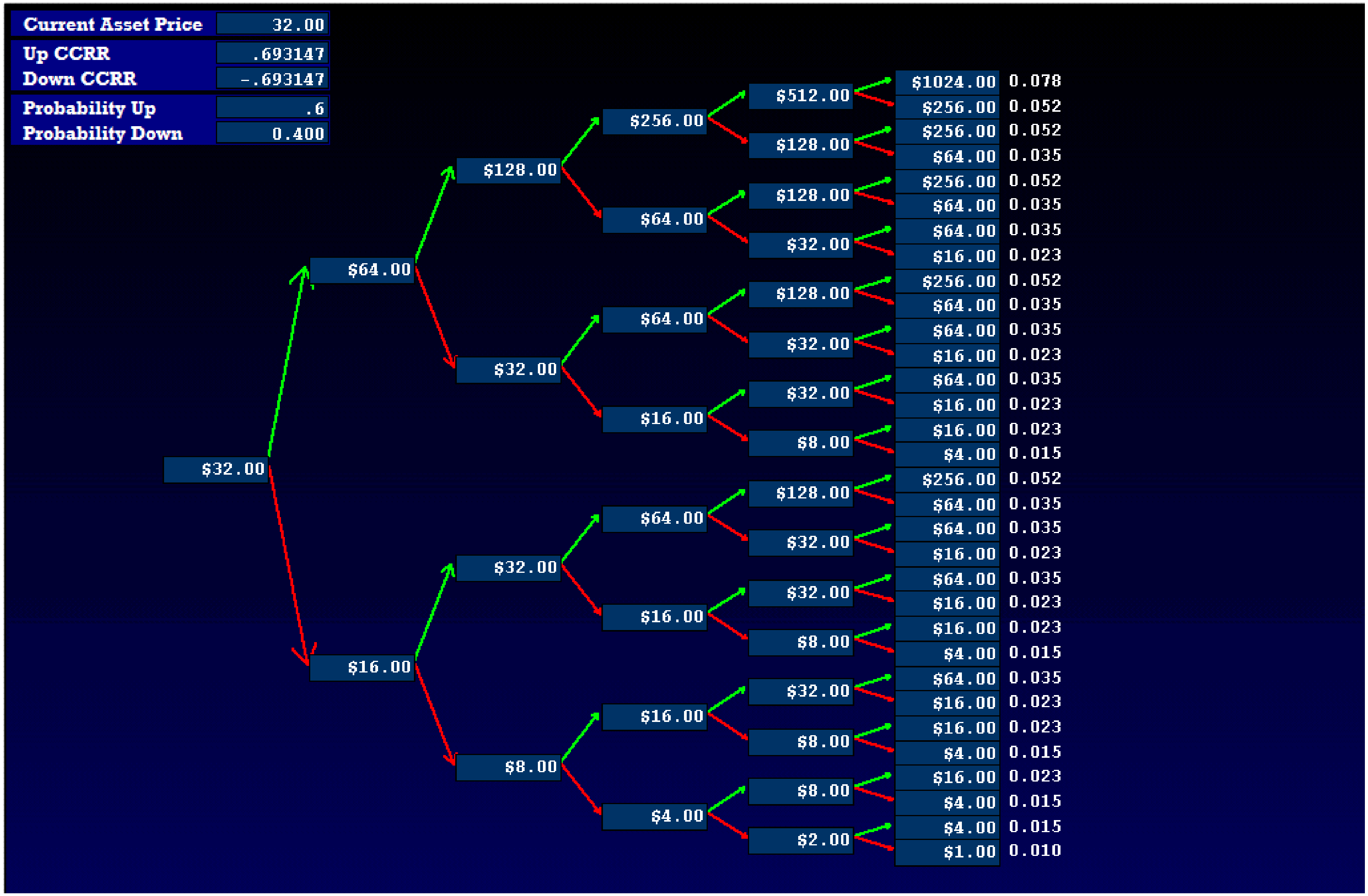
The limited number and pattern of terminal values, however, does not require the up and down rates of return to be of the same magnitude.



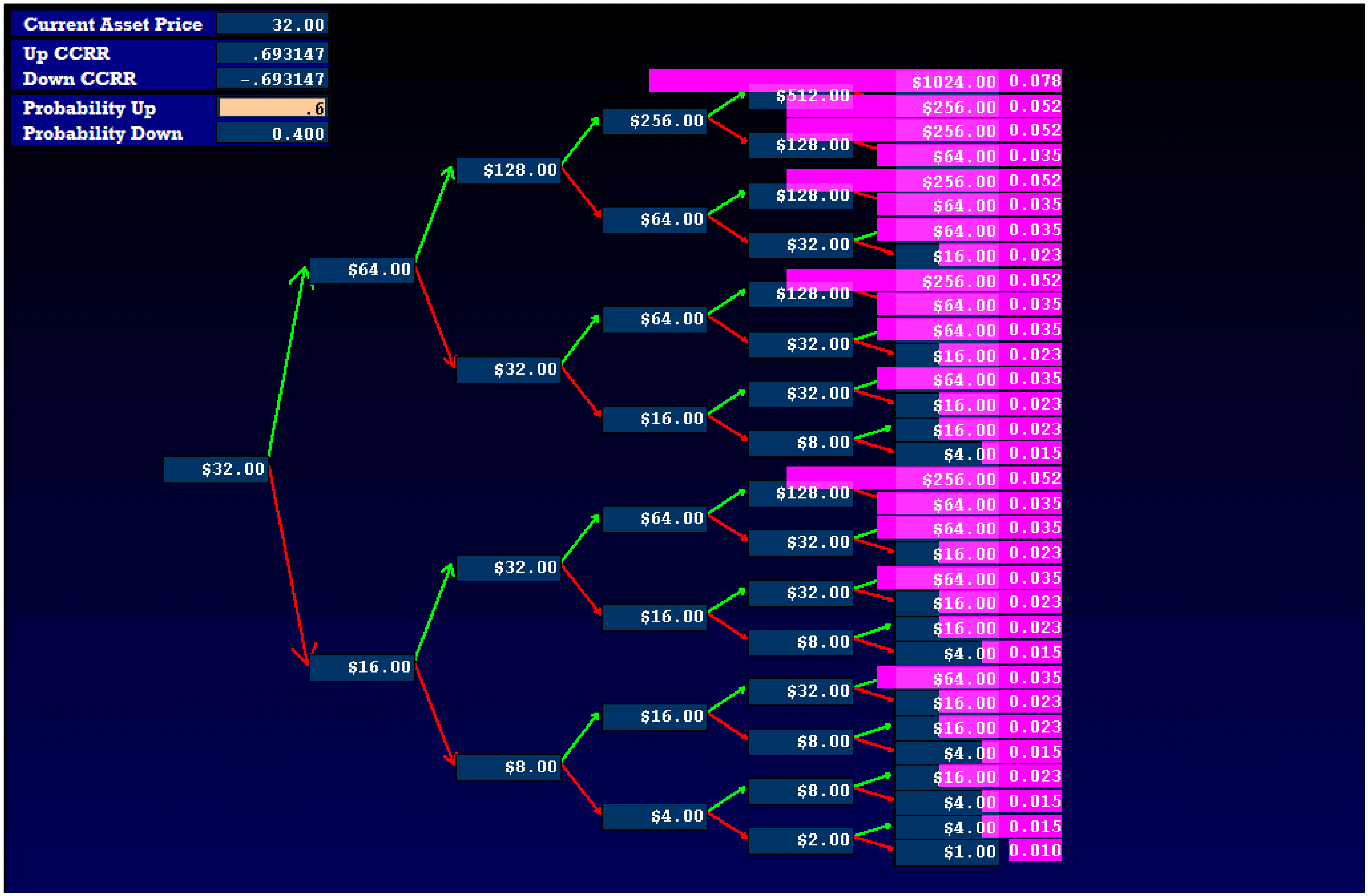
Here we arbitrarily change the down continuously compounded rate of return to -0.287682, a loss of 25% of value. The number of different terminal values we get still follows the pattern of six values with an occurrence pattern of 1, 5, 10, 10, 5, 1.



If the up and down probabilities at each node are both 0.5, then the probability of the stock price following the different paths is all the same. With five steps, the probability of the price following any given path is 1/32 or 0.031.



Here we arbitrarily change the probability of going up at each node to .6. Hence, the probability of the price going to $1,024.00 is now 0.6^5 or 0.078. The probability of the price going to $1.00 is now 0.4^5 or 0.010. The probability of the price following the path that goes up, up, down, up, down is .6 x .6 x .4 x .6 x .4 or 0.035.



With probability bars, we can represent graphically the probability of the stock price going to the different terminal nodes. The width of each bar is proportional to the probability of the stock price going to the terminal node to the immediate left of the bar.

Because price paths in a binomial tree recombine, we need not draw each path separately. Instead, we can draw the potential evolution of stock-price paths as a recombining binomial tree.

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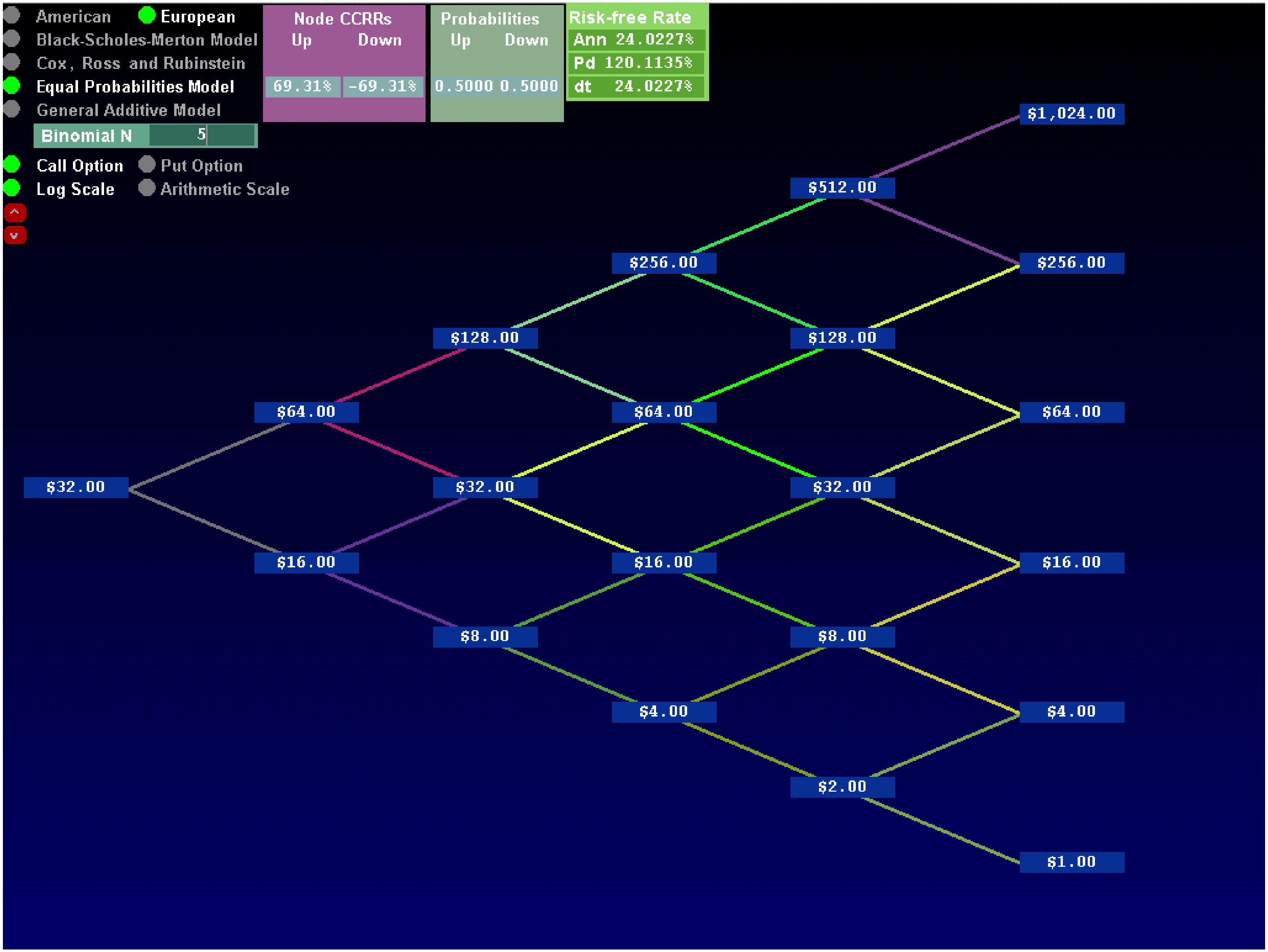
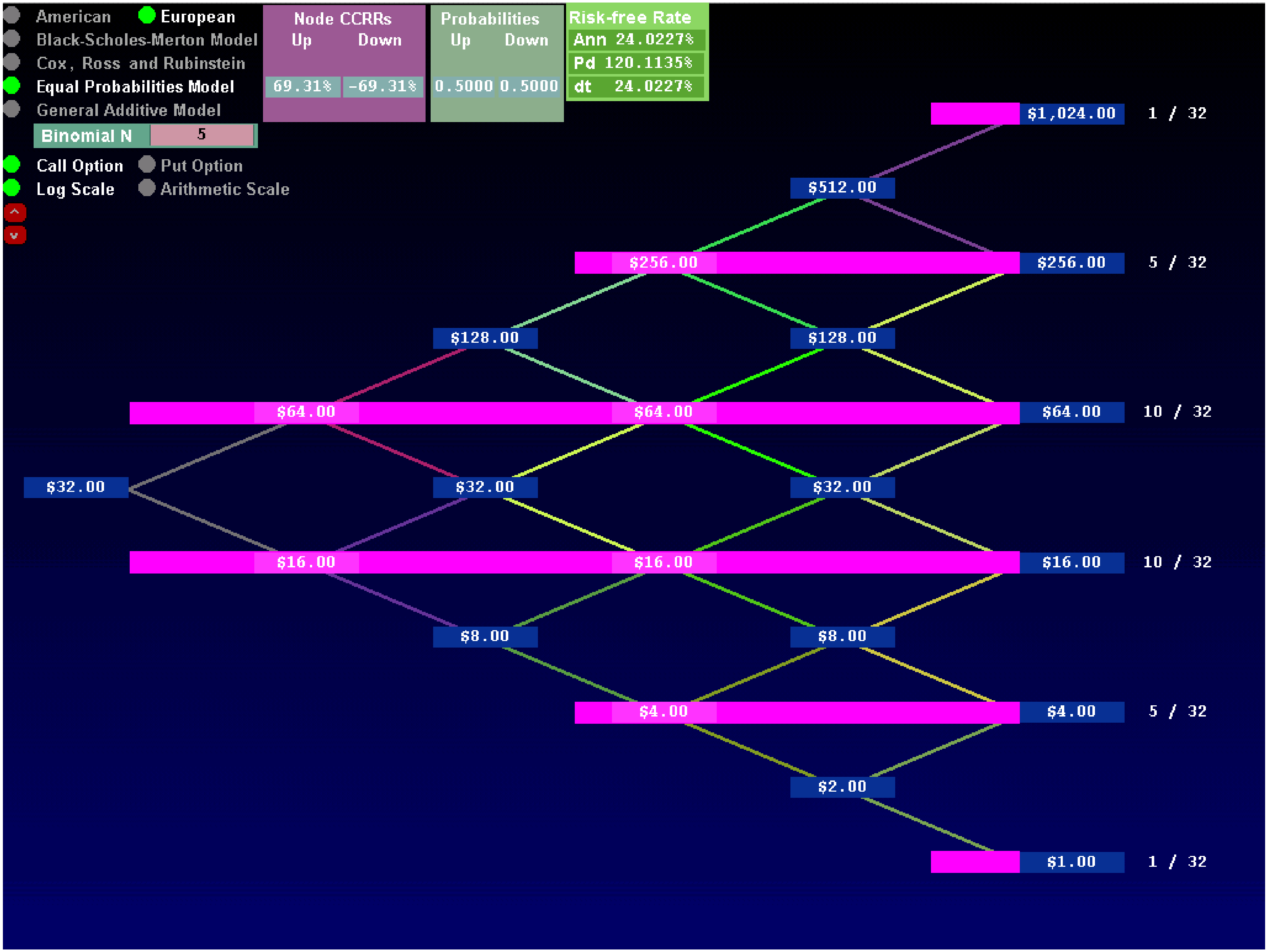


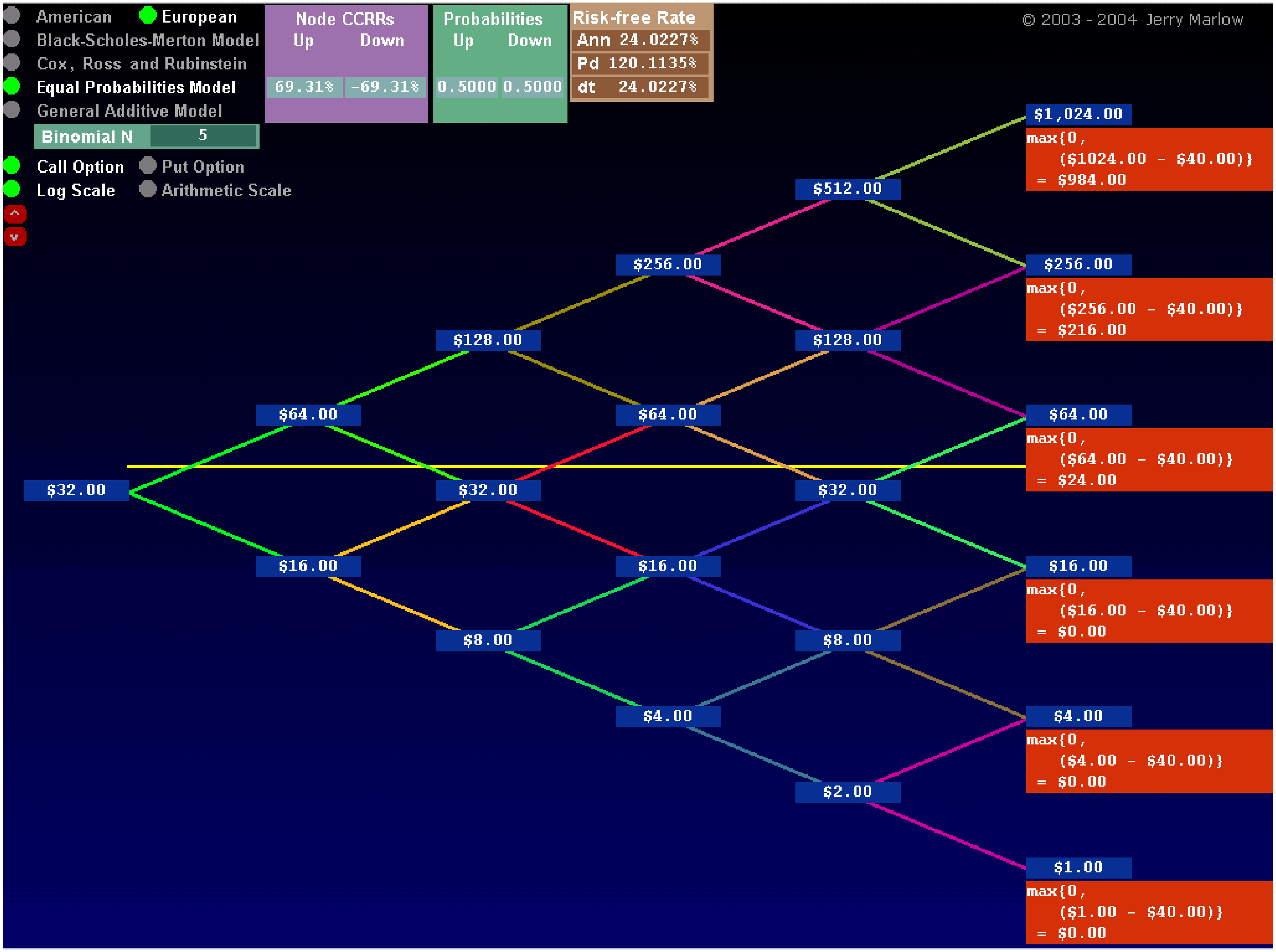
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$32.00
$64.00
$16.00
$128.00
$32.00
$8.00
$256.00
$64.00
$16.00
$4.00
$512.00
$128.00
$32.00
$8.00
$2.00
$1,024.00
$256.00
$64.00
$16.00
$4.00
$1.00

Here we have a recombining tree with the same characteristics as the one we drew initially: The tree has five time steps. At each node, the up continuously compounded rate of return is 69.31%. The down continuously compounded rate of return is -69.31%. The up probability is 0.5. The down probability is 0.5.


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00 1 / 32
$512.00
$256.00
$256.00 5 / 32
$128.00
$128.00
$64.00
$64.00
$64.00 10 / 32
$32.00
$32.00
$32.00
$16.00
$16.00
$16.00 10 / 32
$8.00
$8.00
$4.00
$4.00 5 / 32
$2.00
$1.00 1 / 32

When we draw the tree in this way and, at each node, the probability of the stock price going up is equal to the probability of it going down, we easily can see the probabilities of the stock price getting to each terminal value. The probability bars form a binomial distribution.

Once we have a model that gives us all the potential stock prices at the end of an investment horizon and the probabilities of those prices, it's easy to value European-style and American-style options written on that stock.


American
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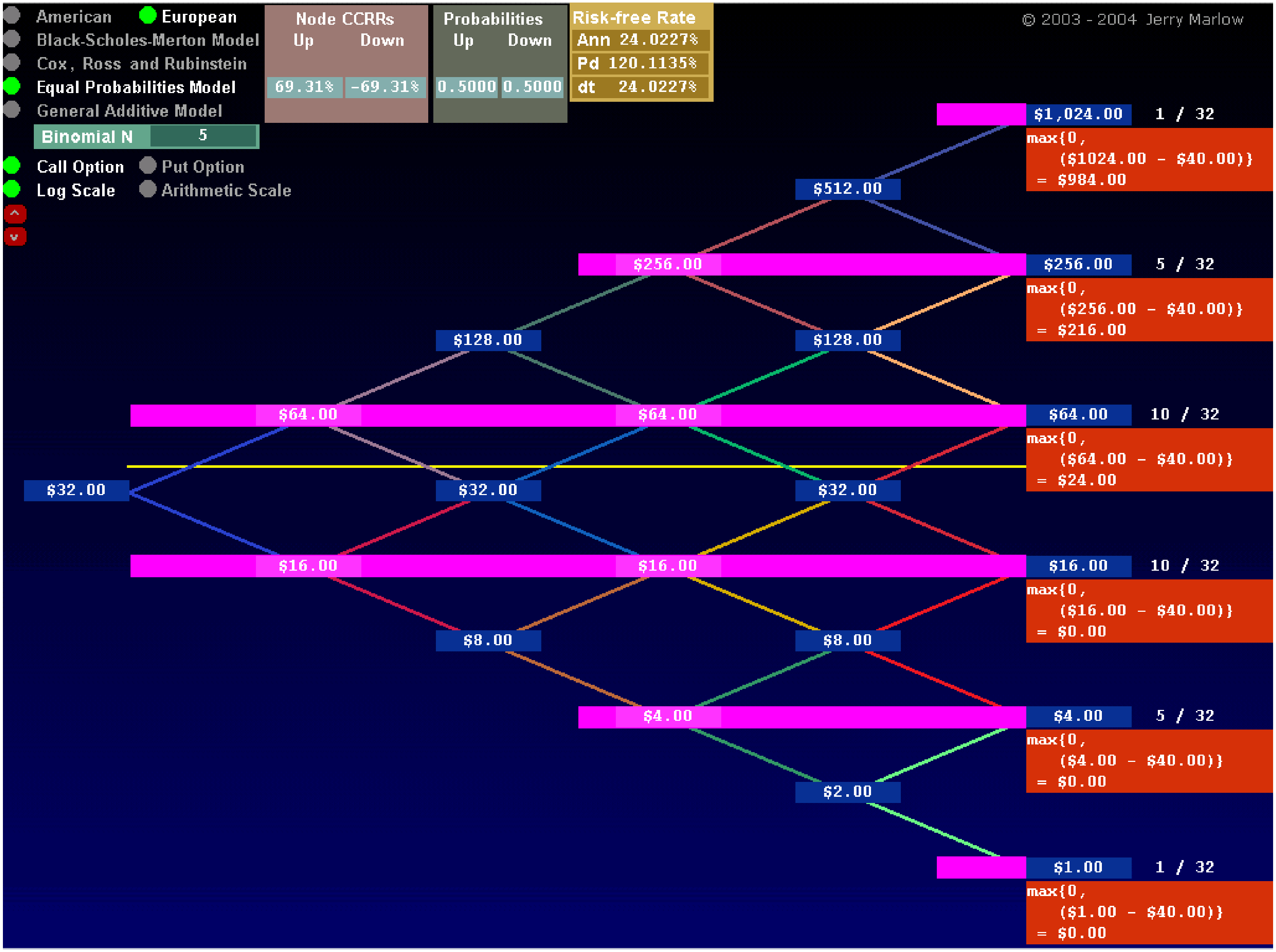
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$32.00
$64.00
$16.00
$128.00
$32.00
$8.00
$256.00
$64.00
$16.00
$4.00
$512.00
$128.00
$32.00
$8.00
$2.00
$1,024.00
max{0, ($1024.00 - $40.00)} = $984.00
$256.00
max{0, ($256.00 - $40.00)} = $216.00
$64.00
max{0, ($64.00 - $40.00)} = $24.00
$16.00
max{0, ($16.00 - $40.00)} = $0.00
$4.00
max{0, ($4.00 - $40.00)} = $0.00
$1.00
max{0, ($1.00 - $40.00)} = $0.00

Recall that a European-style option can be exercised only at the time of its expiration. Here we value a European-style call option that has a strike price of $40.00. The long yellow horizontal line represents the strike price. For a call option, any stock price at or below the yellow line has a payoff of zero.

One way to value a European style option is to follow these steps:

1. Calculate the option payoffs at all the final nodes.

   For a call option, the payoff is the greater of zero or the stock price minus the strike price. The exhibit shows the payoffs of the option at the terminal nodes.


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$1,024.00
1 / 32
max{0, ($1024.00 - $40.00)} = $984.00
$512.00
$256.00
$256.00
5 / 32
max{0, ($256.00 - $40.00)} = $216.00
$128.00
$128.00
$64.00
$64.00
$64.00
10 / 32
max{0, ($64.00 - $40.00)} = $24.00
$32.00
$32.00
$32.00
$16.00
$16.00
$16.00
10 / 32
max{0, ($16.00 - $40.00)} = $0.00
$8.00
$8.00
$4.00
$4.00
5 / 32
max{0, ($4.00 - $40.00)} = $0.00
$2.00
$1.00
1 / 32
max{0, ($1.00 - $40.00)} = $0.00

2. Calculate the probability-weighted values of the payoffs and sum them.

Payoff		Probability		Probability-weighted Value
$984.00	x	1/32	=	$30.75
$216.00	x	5/32	=	$33.75
$24.00	x	10/32	=	$7.50
$0.00	x	10/32	=	$0.00
$0.00	x	5/32	=	$0.00
$0.00	x	1/32	=	$0.00
		Total:		$72.00

3. For the option's time to expiration, discount this probability-weighted future value at the continuously compounded risk-free rate.

   In this example, we use an unrealistically high annual risk-free rate of 24.0227% which, for the five years to expiration, translates into a rate of 120.1135% for the investment period.

   Option value = $72.00/exp(1.201135)
   = $21.66

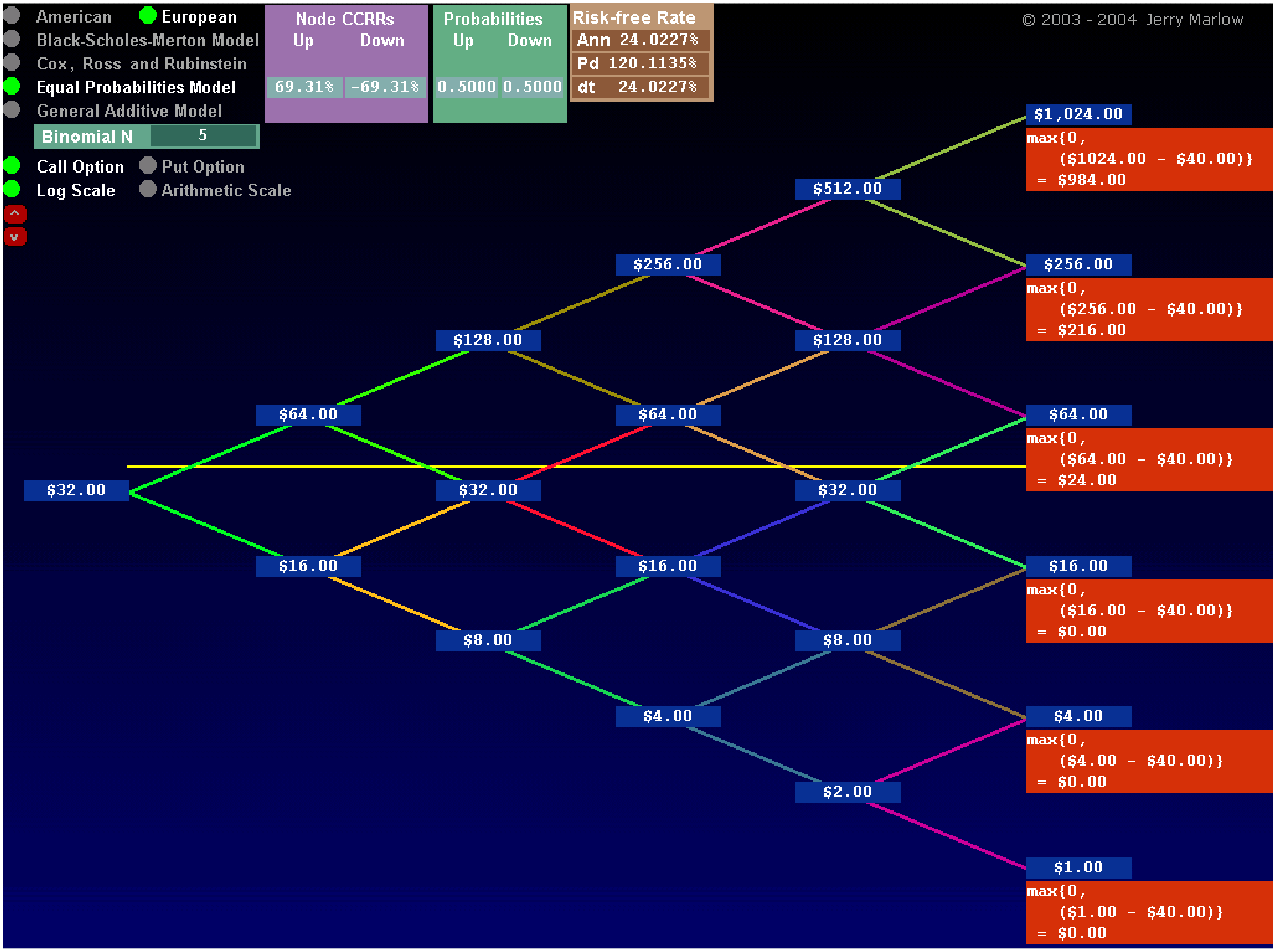
This method of valuing a European-style call option is similar to the method we used initially with the risk-neutral, geometric-Brownian-motion model, which was:

Find the possible end-of-period stock prices.

Calculate the payoff that each stock price produces.

Find the probability-weighted present values of the payoffs.

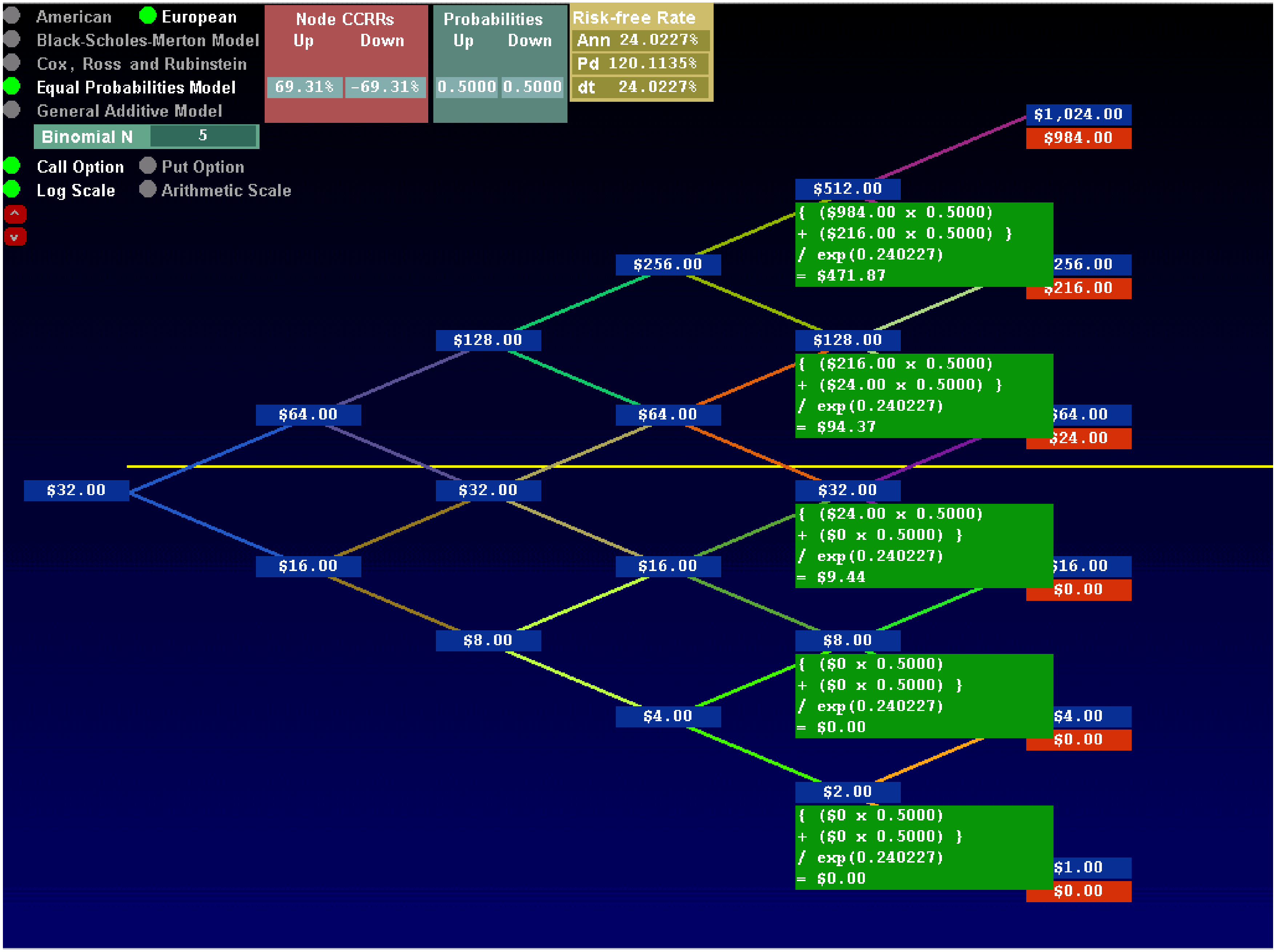
Sum them.


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$32.00
$64.00
$16.00
$128.00
$32.00
$8.00
$256.00
$64.00
$16.00
$4.00
$512.00
$128.00
$32.00
$8.00
$2.00
$1,024.00
max{0, ($1024.00 – $40.00)} = $984.00
$256.00
max{0, ($256.00 – $40.00)} = $216.00
$64.00
max{0, ($64.00 – $40.00)} = $24.00
$16.00
max{0, ($16.00 – $40.00)} = $0.00
$4.00
max{0, ($4.00 – $40.00)} = $0.00
$1.00
max{0, ($1.00 – $40.00)} = $0.00

Backward induction:
Walking values back down the tree

An alternative way to value a European style option is to follow a set of steps that starts out the same, but then changes. We use the alternative method to value the same stock-price forecast and the same option.

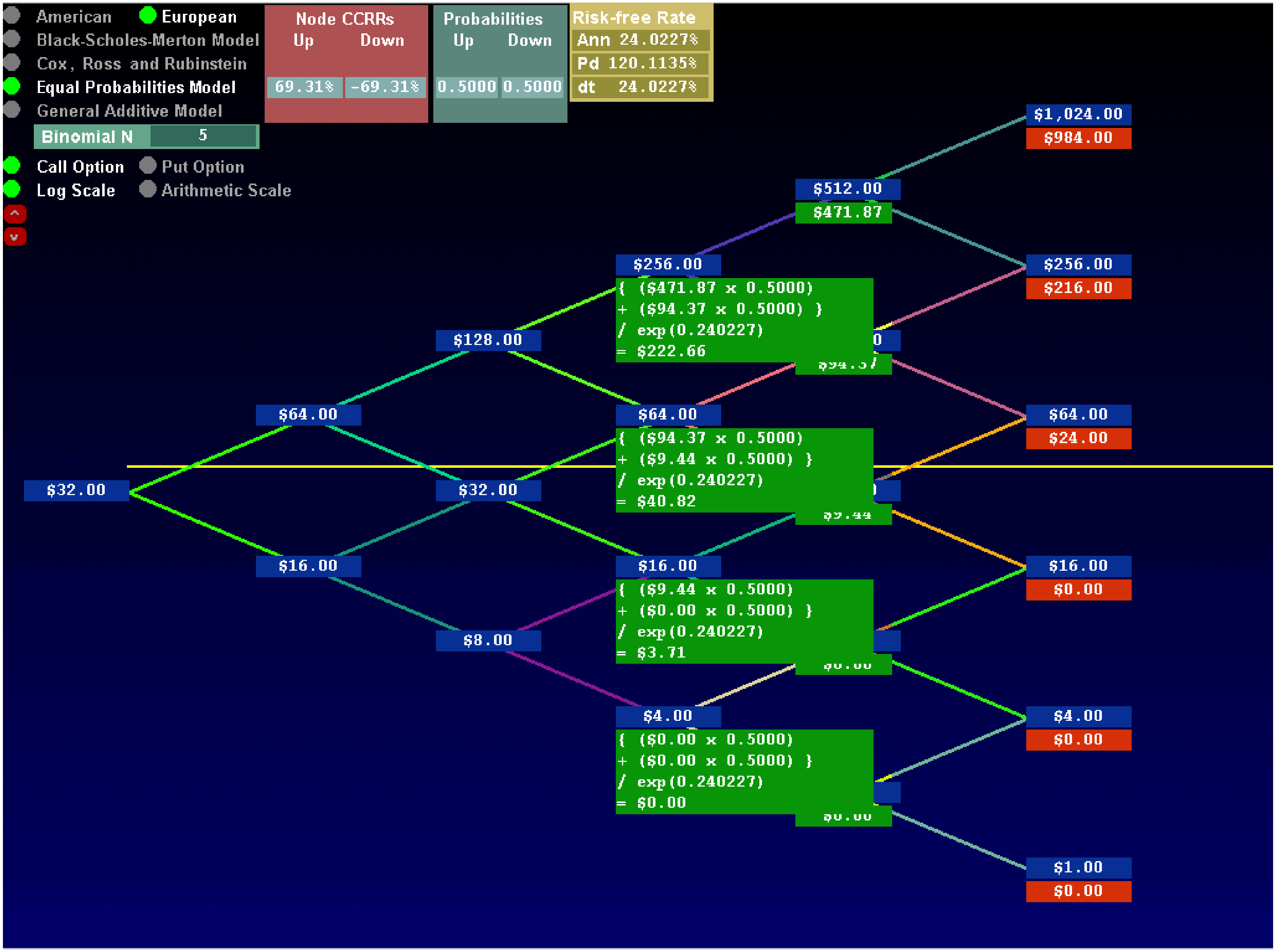
1. Calculate the option payoffs at all the final nodes.


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00
$984.00
$512.00
{ ($984.00 x 0.5000)
+ ($216.00 x 0.5000) }
/ exp(0.240227)
= $471.87
$256.00
$256.00
$216.00
$128.00
$128.00
{ ($216.00 x 0.5000)
+ ($24.00 x 0.5000) }
/ exp(0.240227)
= $94.37
$64.00
$64.00
$64.00
$24.00
$32.00
$32.00
$32.00
{ ($24.00 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $9.44
$16.00
$16.00
$16.00
$0.00
$8.00
$8.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$4.00
$4.00
$0.00
$2.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$1.00
$0.00

2. At each previous node, calculate the probability-weighted, point-in-time value of the subsequent payoffs.

   From the vantage point of the $512.00 stock-price node, the probability of a payoff of $984.00 is 0.5; the probability of a payoff of $216.00 is 0.5. The risk-free rate for a period of time equal to one time step is 0.240227. Hence, at the $512.00 stock-price node, the probability-weighted, point-in-time value of the subsequent payoffs is $471.87.

   We make the corresponding calculations at the other nodes in this stack.

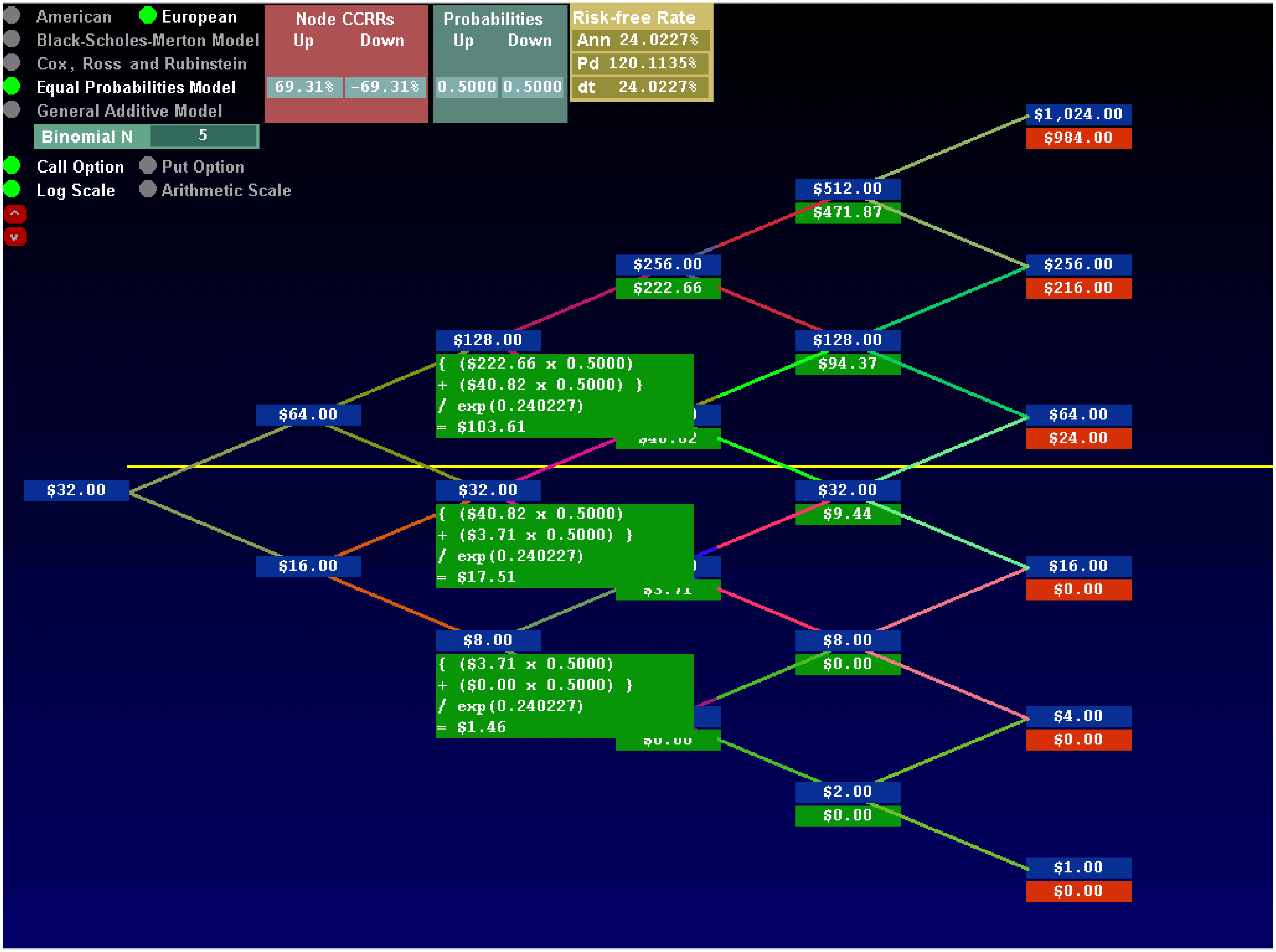

American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$32.00
$64.00
$16.00
$128.00
$32.00
$8.00
$256.00
{ ($471.87 x 0.5000)
+ ($94.37 x 0.5000) }
/ exp(0.240227)
= $222.66
$64.00
{ ($94.37 x 0.5000)
+ ($9.44 x 0.5000) }
/ exp(0.240227)
= $40.82
$16.00
{ ($9.44 x 0.5000)
+ ($0.00 x 0.5000) }
/ exp(0.240227)
= $3.71
$4.00
{ ($0.00 x 0.5000)
+ ($0.00 x 0.5000) }
/ exp(0.240227)
= $0.00
$512.00
$471.87
$1,024.00
$984.00
$256.00
$216.00
$64.00
$24.00
$16.00
$0.00
$4.00
$0.00
$1.00
$0.00

3. At each previous node, calculate the probability-weighted, point-in-time value of the subsequent values.

   From the vantage point of the $128.00 stock-price node, the probability of a subsequent point-in-time value of $471.87 is 0.5; the probability of a value of $94.37 is 0.5. The one-time-step risk-free rate is 0.240227. Hence, at the $128.00 stock-price node, the probability-weighted, point-in-time value of the subsequent payoffs is $222.26.

   We make the same calculations at the other nodes in this stack.

   And thus, through a process of backward induction, we work our way back down the tree to the present time.


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
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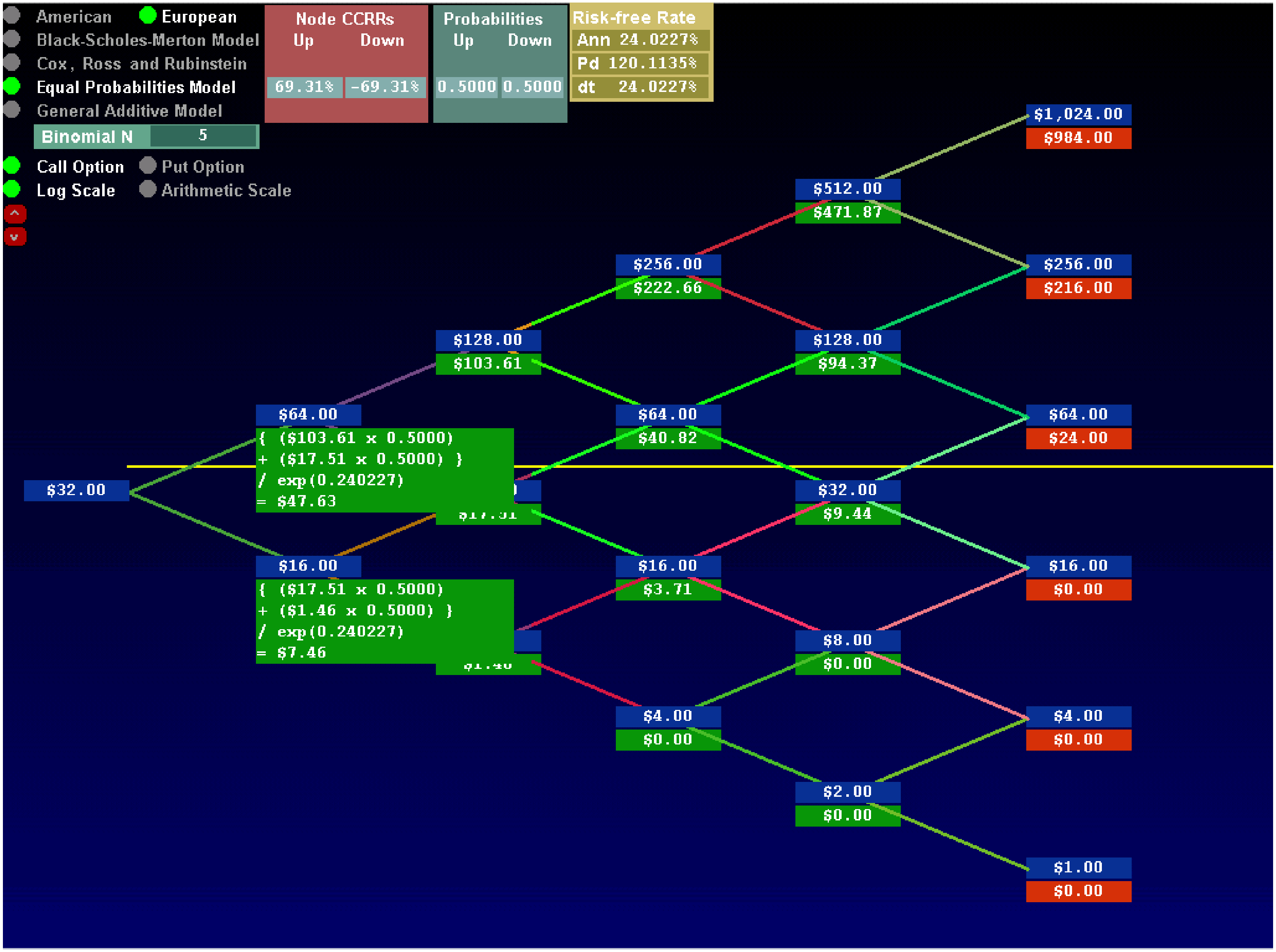
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$32.00
$64.00
$16.00
$128.00
{ ($222.66 x 0.5000)
+ ($40.82 x 0.5000) }
/ exp(0.240227)
= $103.61
$32.00
{ ($40.82 x 0.5000)
+ ($3.71 x 0.5000) }
/ exp(0.240227)
= $17.51
$8.00
{ ($3.71 x 0.5000)
+ ($0.00 x 0.5000) }
/ exp(0.240227)
= $1.46
$256.00
$222.66
$512.00
$471.87
$128.00
$94.37
$32.00
$9.44
$8.00
$0.00
$2.00
$0.00
$1,024.00
$984.00
$256.00
$216.00
$64.00
$24.00
$16.00
$0.00
$4.00
$0.00
$1.00
$0.00



American
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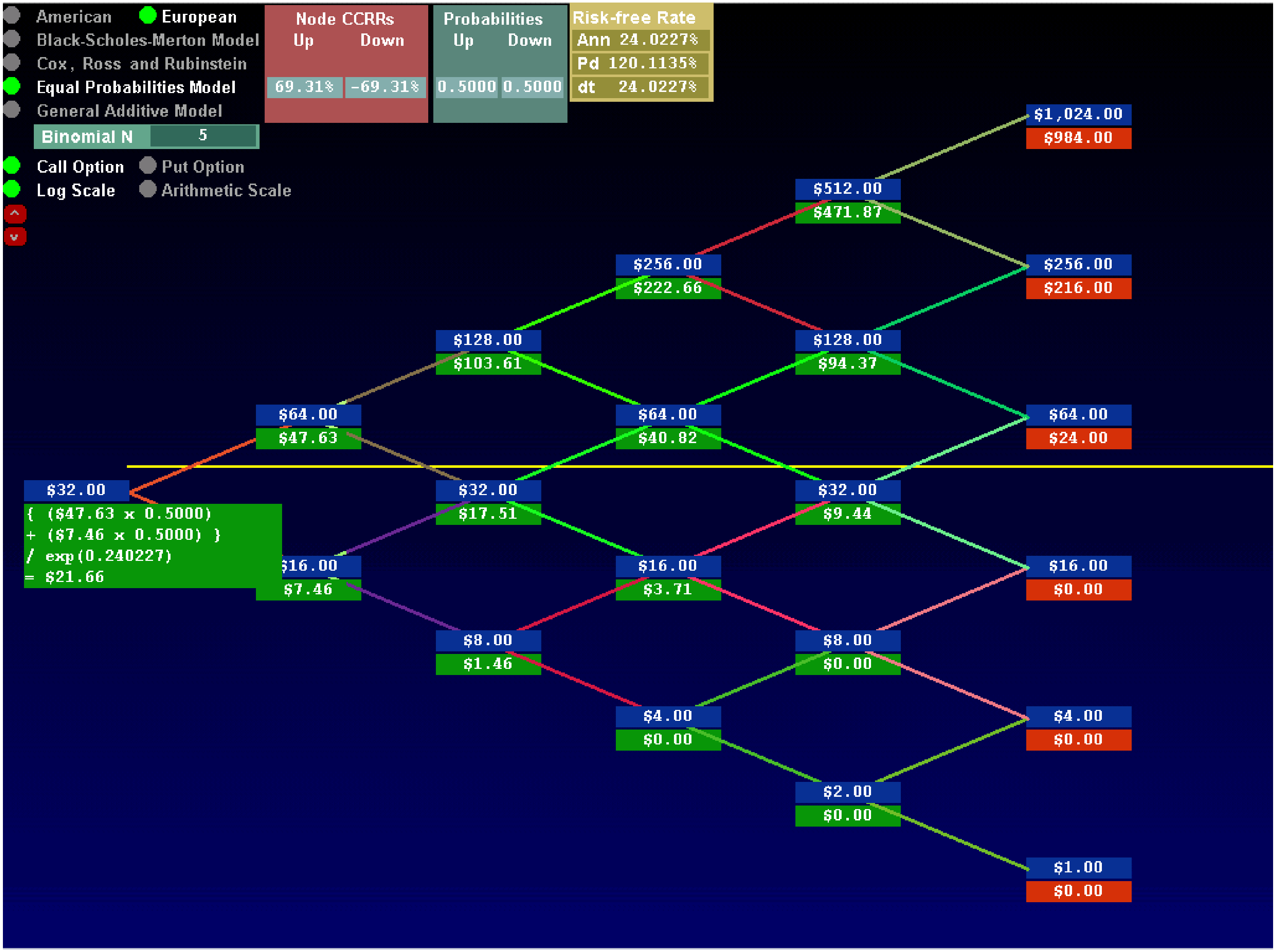
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00
$984.00
$512.00
$471.87
$256.00
$222.66
$256.00
$216.00
$128.00
$103.61
$128.00
$94.37
$64.00
$40.82
$64.00
$24.00
$64.00
{ ($103.61 x 0.5000)
+ ($17.51 x 0.5000) }
/ exp(0.240227)
= $47.63
$32.00
$32.00
$9.44
$16.00
$3.71
$16.00
$0.00
$16.00
{ ($17.51 x 0.5000)
+ ($1.46 x 0.5000) }
/ exp(0.240227)
= $7.46
$8.00
$0.00
$4.00
$0.00
$4.00
$0.00
$2.00
$0.00
$1.00
$0.00


American
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Arithmetic Scale
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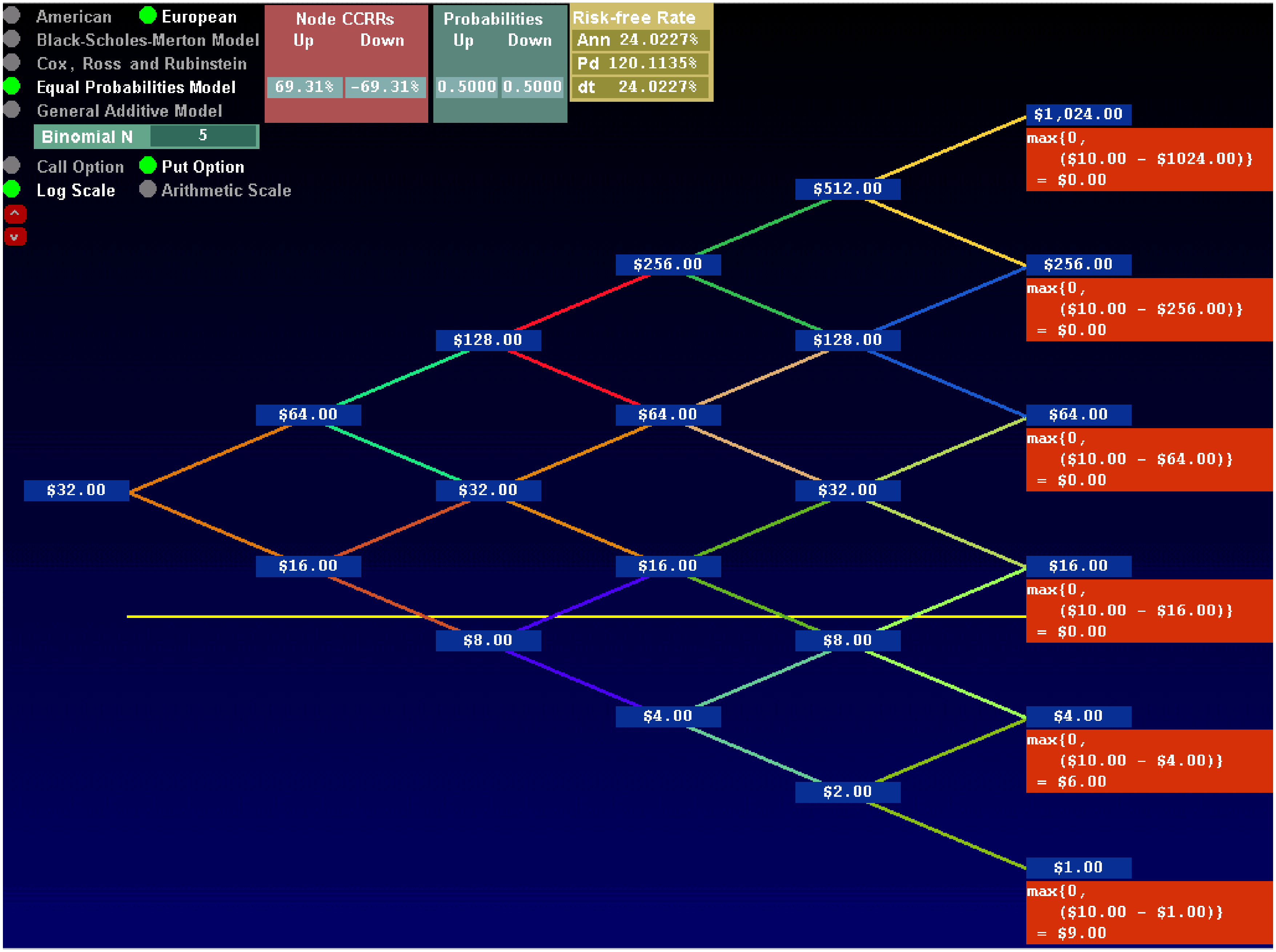
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00
$984.00
$512.00
$471.87
$256.00
$222.66
$256.00
$216.00
$128.00
$103.61
$128.00
$94.37
$64.00
$47.63
$64.00
$40.82
$64.00
$24.00
$32.00
{ ($47.63 x 0.5000)
+ ($7.46 x 0.5000) }
/ exp(0.240227)
= $21.66
$32.00
$17.51
$32.00
$9.44
$16.00
$7.46
$16.00
$3.71
$16.00
$0.00
$8.00
$1.46
$8.00
$0.00
$4.00
$0.00
$4.00
$0.00
$2.00
$0.00
$1.00
$0.00

Using backward induction to walk the values back down the tree gives us a probability-weighted present value of $21.66. This is the value of this European-style call option on a stock that pays no dividends valued on this tree. It is the same value we got when we simply calculated the probabilities of the payoffs and found their probability-weighted present value.

Using backward induction to value a European-style put option

To value a European-style put option using backward induction, we follow the same sequence of steps. The only difference is that the payoff of a put is the greater of zero or the strike price minus the stock price.

As an example, we use the same forecast and five-year time to expiration as before. The put has a strike price of $10.00.


American
European
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General Additive Model
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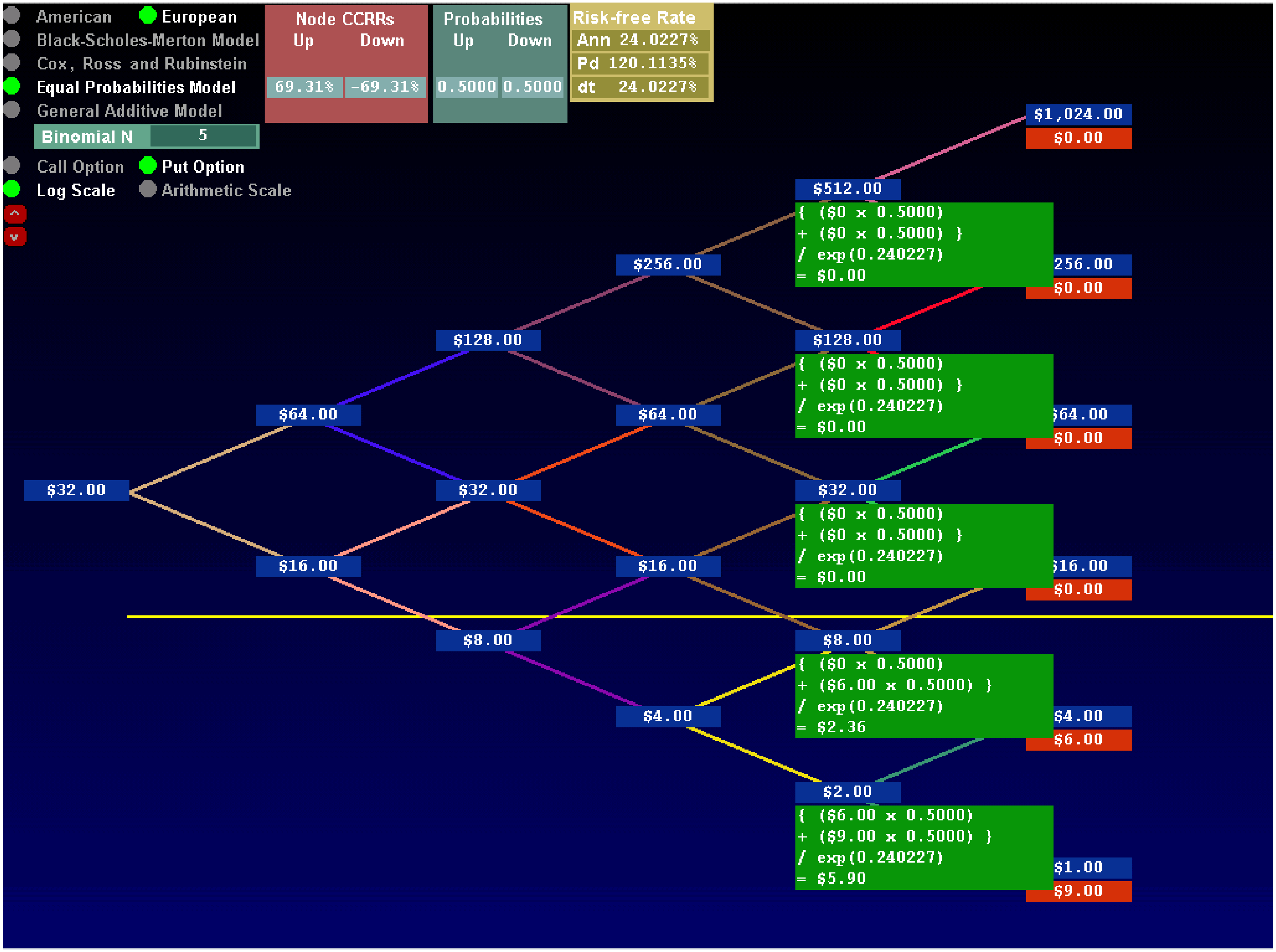
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$32.00
$64.00
$16.00
$128.00
$32.00
$8.00
$256.00
$64.00
$16.00
$4.00
$512.00
$128.00
$32.00
$8.00
$2.00
$1,024.00
max{0, ($10.00 - $1024.00)} = $0.00
$256.00
max{0, ($10.00 - $256.00)} = $0.00
$64.00
max{0, ($10.00 - $64.00)} = $0.00
$16.00
max{0, ($10.00 - $16.00)} = $0.00
$4.00
max{0, ($10.00 - $4.00)} = $6.00
$1.00
max{0, ($10.00 - $1.00)} = $9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
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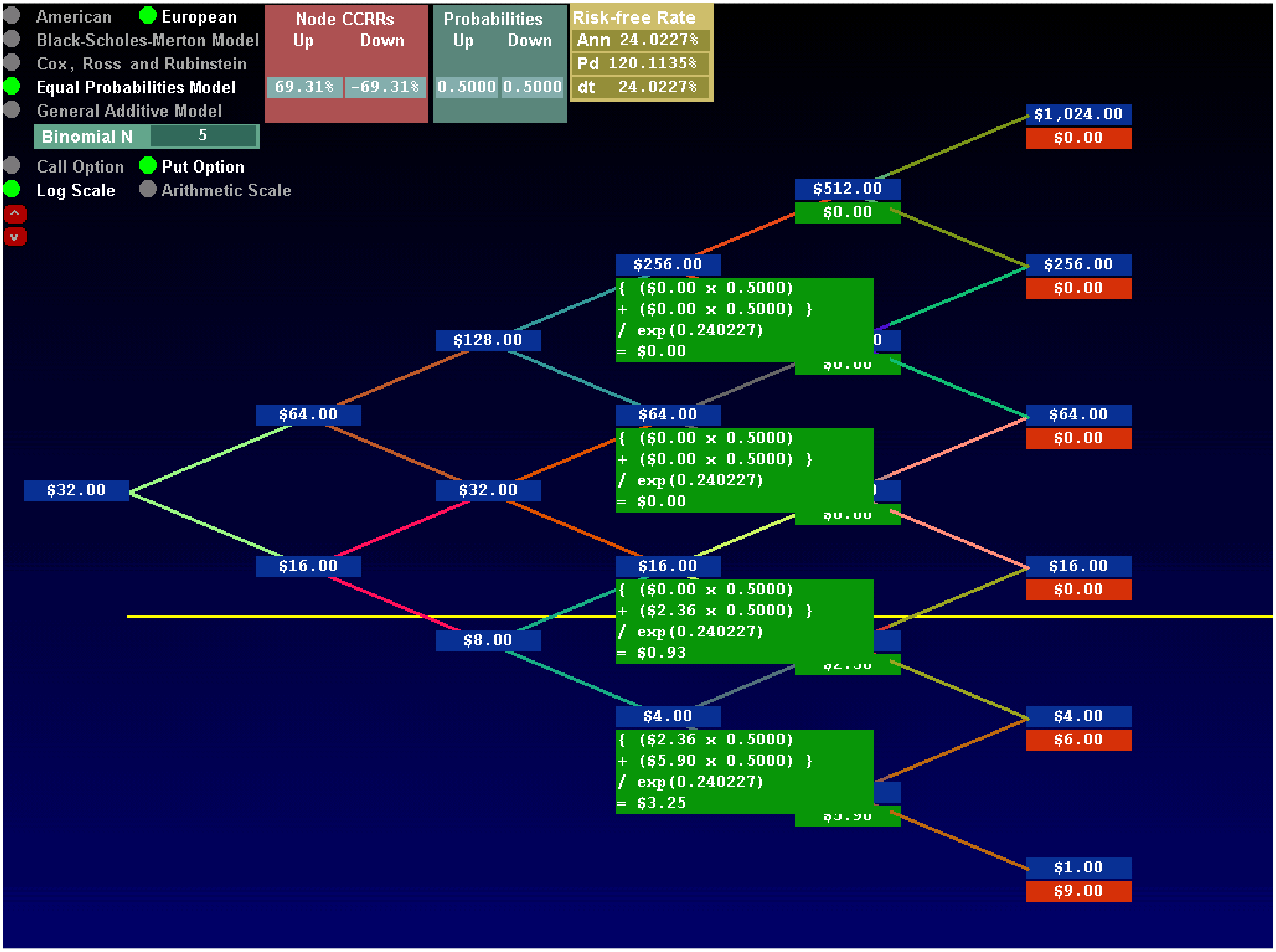
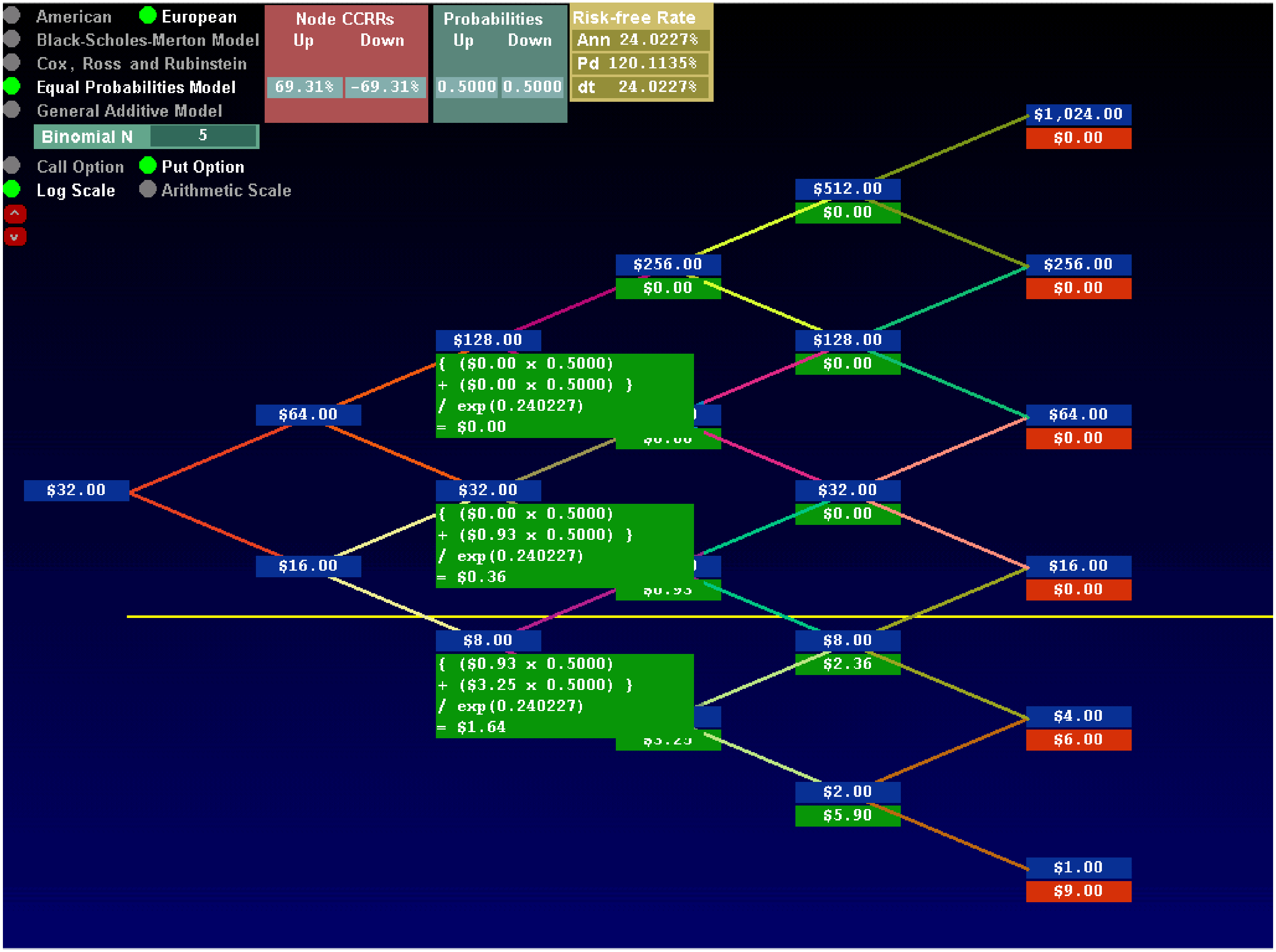
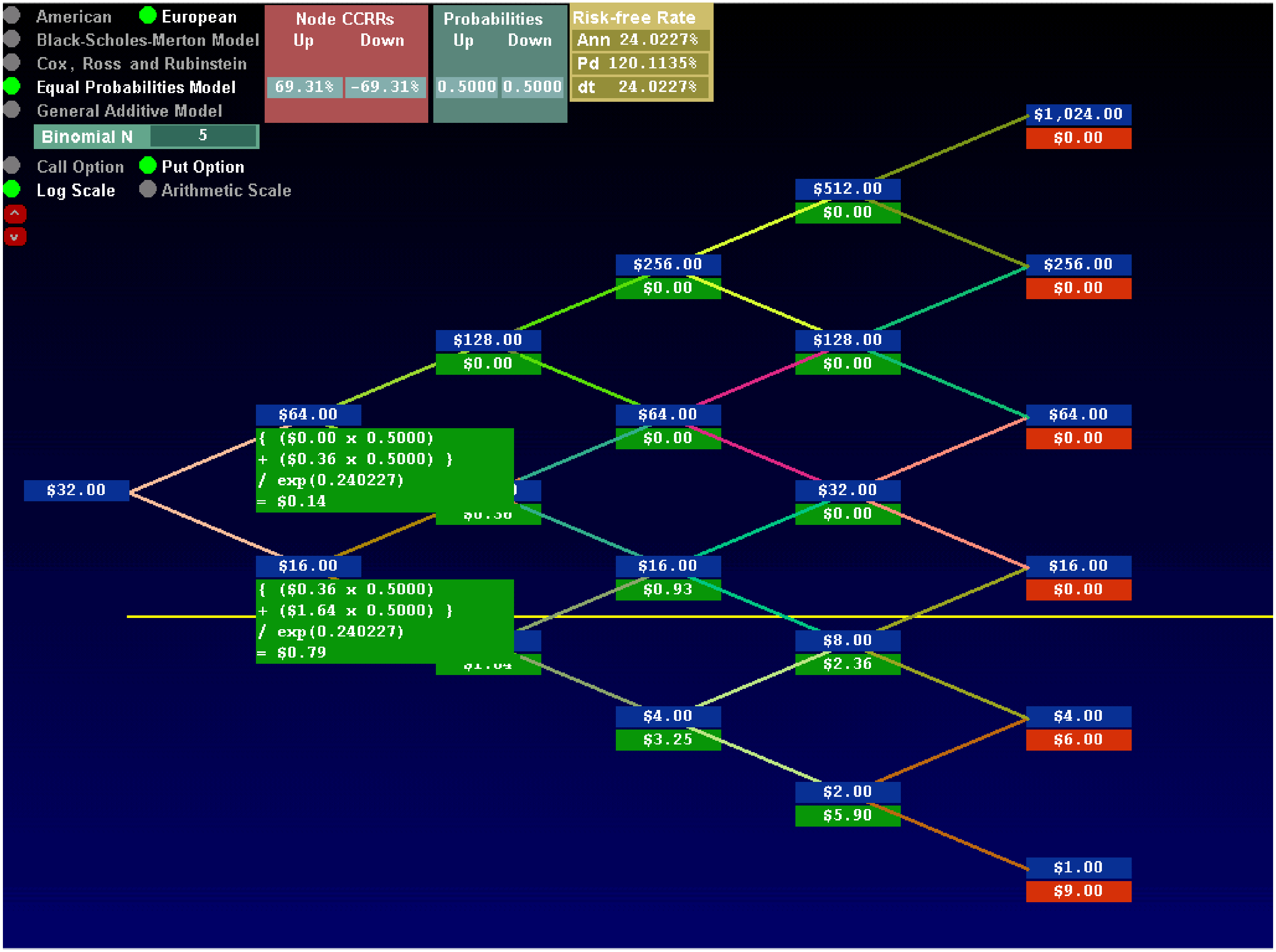
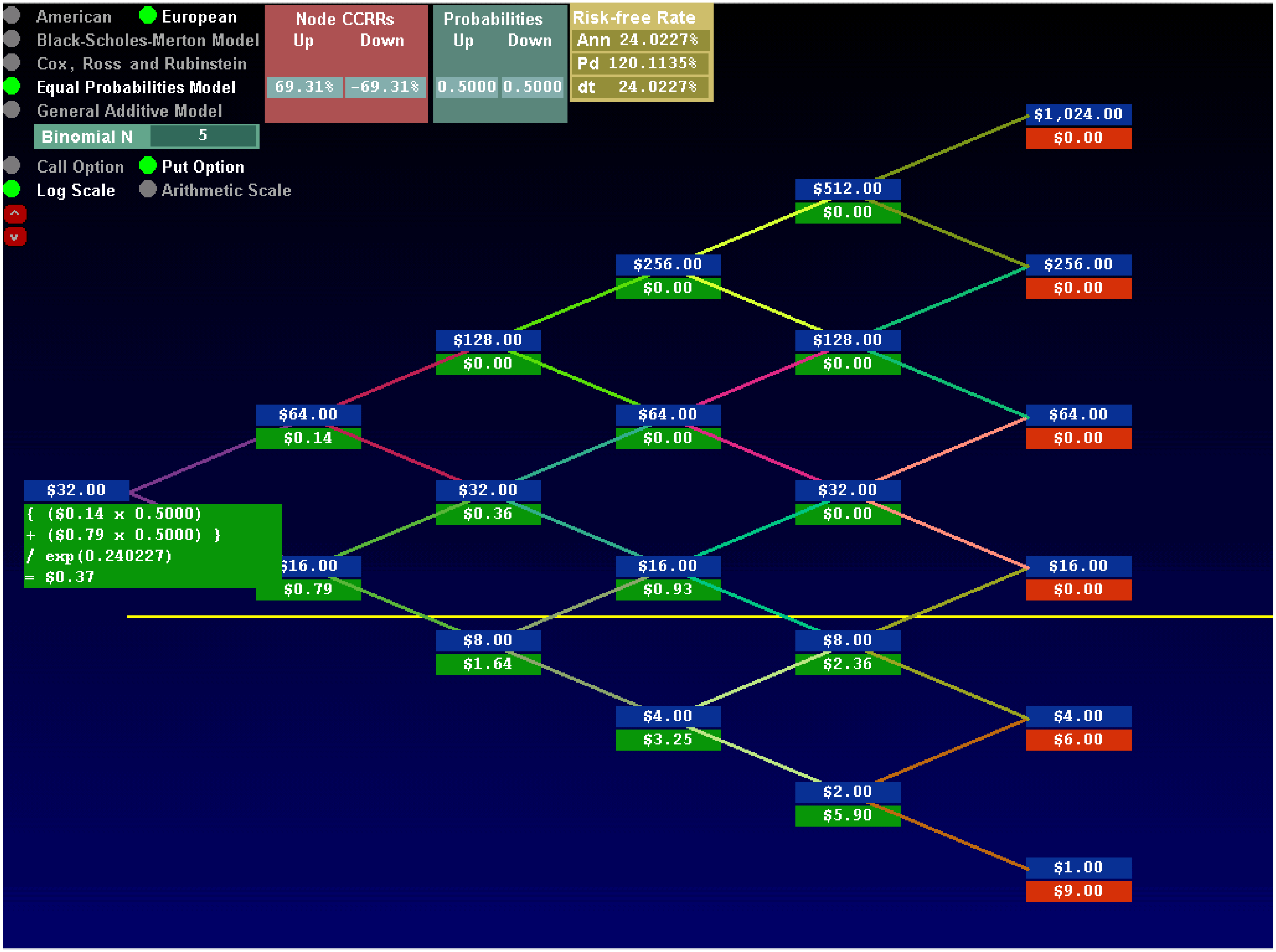
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00
$0.00
$512.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$256.00
$0.00
$128.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$64.00
$0.00
$32.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$16.00
$0.00
$8.00
{ ($0 x 0.5000)
+ ($6.00 x 0.5000) }
/ exp(0.240227)
= $2.36
$4.00
$6.00
$2.00
{ ($6.00 x 0.5000)
+ ($9.00 x 0.5000) }
/ exp(0.240227)
= $5.90
$1.00
$9.00
$256.00
$128.00
$64.00
$32.00
$16.00
$8.00
$4.00
$32.00



American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00
$0.00
$512.00
$0.00
$256.00
{ ($0.00 x 0.5000)
+ ($0.00 x 0.5000) }
/ exp(0.240227)
= $0.00
$256.00
$0.00
$128.00
$64.00
{ ($0.00 x 0.5000)
+ ($0.00 x 0.5000) }
/ exp(0.240227)
= $0.00
$64.00
$0.00
$32.00
$32.00
$16.00
{ ($0.00 x 0.5000)
+ ($2.36 x 0.5000) }
/ exp(0.240227)
= $0.93
$16.00
$0.00
$8.00
$4.00
{ ($2.36 x 0.5000)
+ ($5.90 x 0.5000) }
/ exp(0.240227)
= $3.25
$4.00
$6.00
$1.00
$9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00
$0.00
$512.00
$0.00
$256.00
$0.00
$256.00
$0.00
$128.00
{ ($0.00 x 0.5000)
+ ($0.00 x 0.5000) }
/ exp(0.240227)
= $0.00
$128.00
$0.00
$64.00
$64.00
$0.00
$32.00
$32.00
{ ($0.00 x 0.5000)
+ ($0.93 x 0.5000) }
/ exp(0.240227)
= $0.36
$32.00
$0.00
$16.00
$16.00
$0.00
$8.00
{ ($0.93 x 0.5000)
+ ($3.25 x 0.5000) }
/ exp(0.240227)
= $1.64
$8.00
$2.36
$4.00
$6.00
$2.00
$5.90
$1.00
$9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00
$0.00
$512.00
$0.00
$256.00
$0.00
$256.00
$0.00
$128.00
$0.00
$128.00
$0.00
$64.00
$64.00
$0.00
$64.00
$0.00
{ ($0.00 x 0.5000)
+ ($0.36 x 0.5000) }
/ exp(0.240227)
= $0.14
$32.00
$32.00
$0.00
$16.00
$16.00
$0.93
$16.00
$0.00
{ ($0.36 x 0.5000)
+ ($1.64 x 0.5000) }
/ exp(0.240227)
= $0.79
$8.00
$2.36
$4.00
$3.25
$4.00
$6.00
$2.00
$5.90
$1.00
$9.00

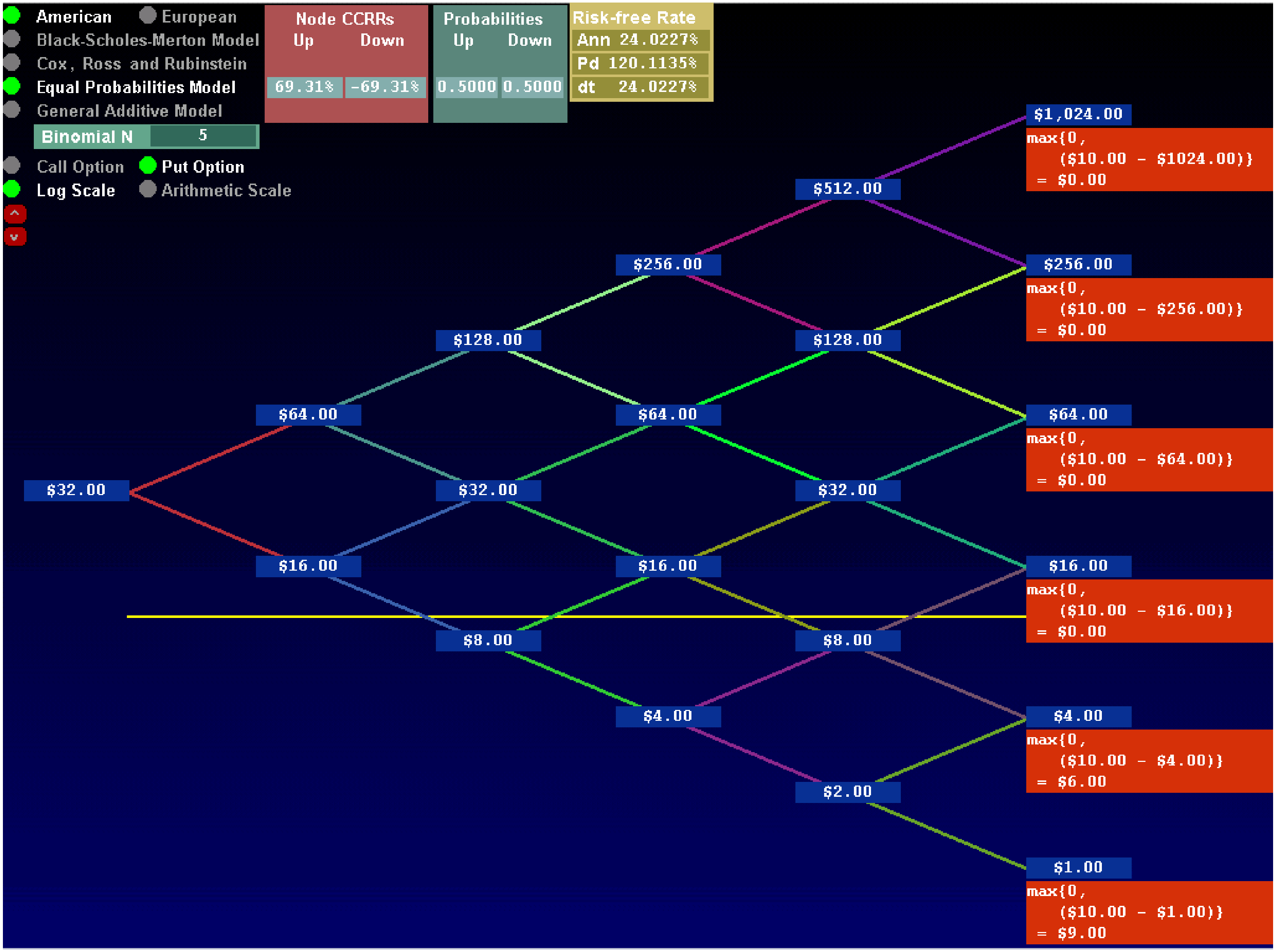

American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$32.00
{ ($0.14 x 0.5000)
+ ($0.79 x 0.5000) }
/ exp(0.240227)
= $0.37
$64.00
$0.14
$16.00
$0.79
$128.00
$0.00
$32.00
$0.36
$8.00
$1.64
$256.00
$0.00
$64.00
$0.00
$16.00
$0.93
$4.00
$3.25
$512.00
$0.00
$128.00
$0.00
$32.00
$0.00
$8.00
$2.36
$2.00
$5.90
$1,024.00
$0.00
$256.00
$0.00
$64.00
$0.00
$16.00
$0.00
$4.00
$6.00
$1.00
$9.00

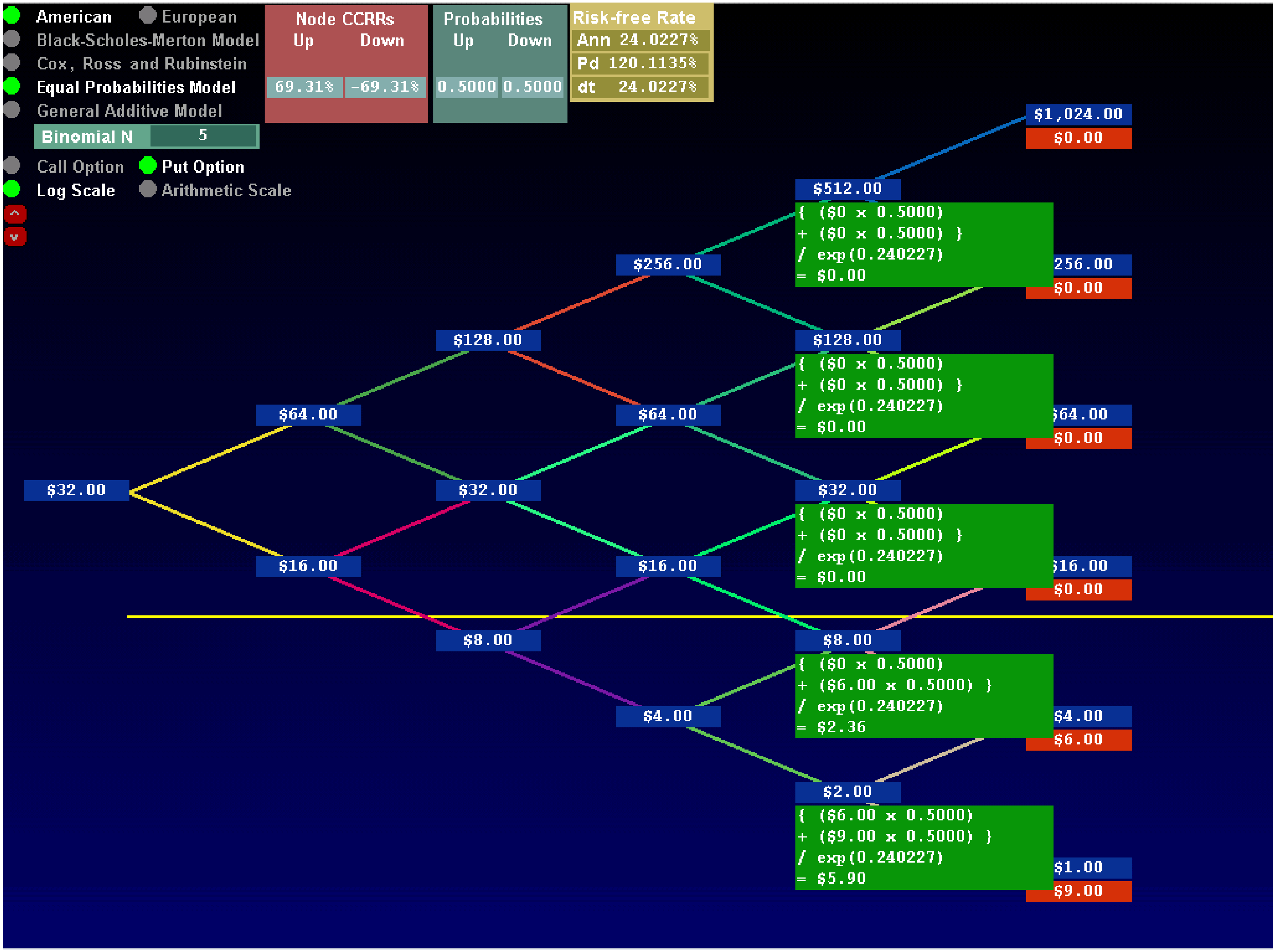
Using this model, this number of time steps, and this forecast for the underlying stock, the value of this European-style put option with a strike price of $10.00 is $0.37.

Using backward induction to value an American-style put option

A European-style option can be exercised only at the time of its expiration. An American-style option can be exercised at any time up until and at its time of expiration. In a well-behaved model, it is never advantageous to exercise early an American-style call option on a stock that pays no dividends. Hence, well-behaved models give the same values for European- and American-style call options on stocks that pay no dividends. It may, however, be advantageous to exercise an American-style put prior to its expiration.

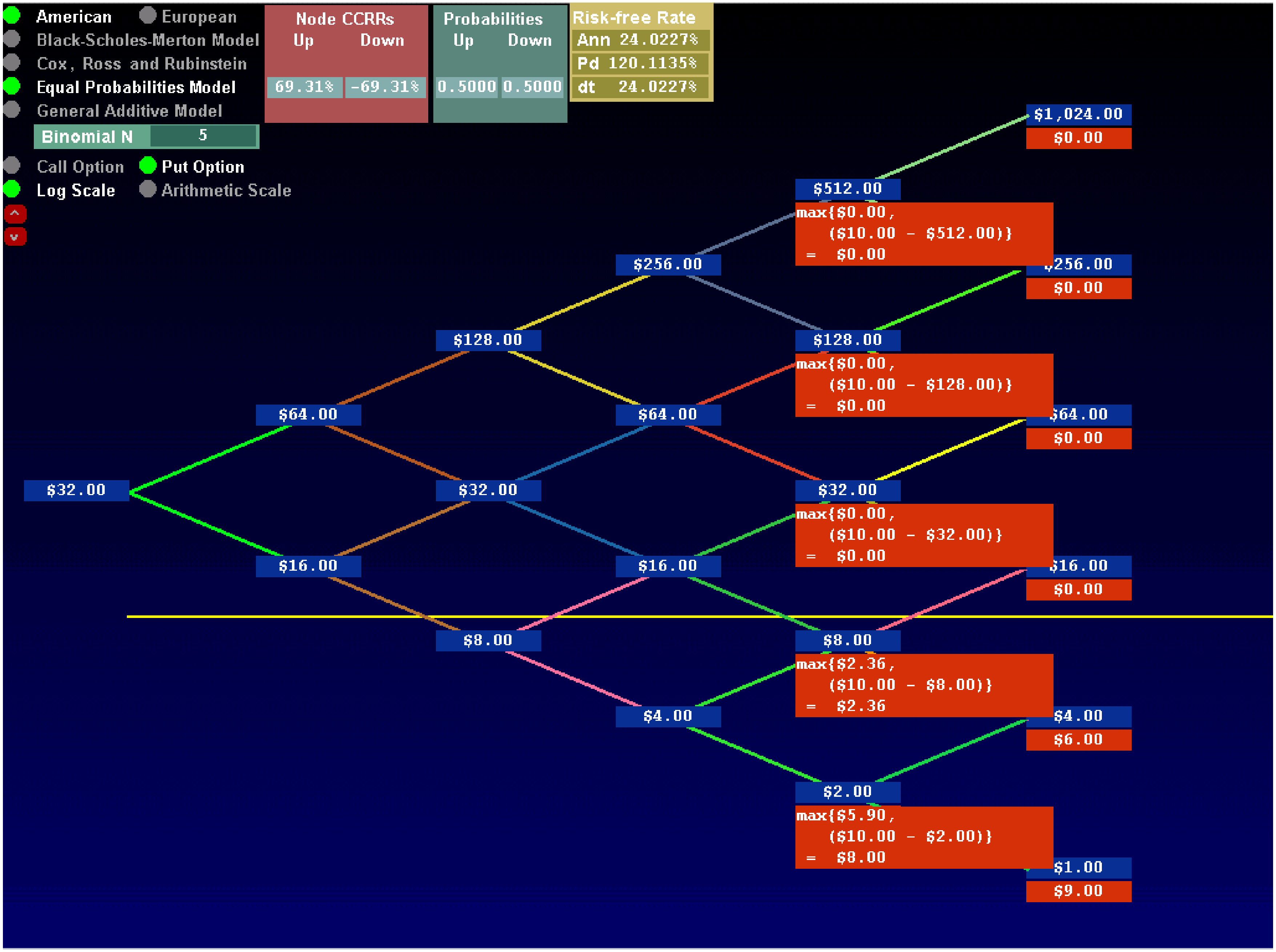
In valuing American-style options with binomial models, one can take into account the value that the privilege of early exercise adds. To value an American-style put, we begin the same way we did with the European-style put: We calculate the option payoffs at the terminal nodes. Then, at each previous node, we calculate the probability-weighted, point-in-time value of the subsequent payoffs.


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$32.00
$64.00
$16.00
$128.00
$32.00
$8.00
$256.00
$64.00
$16.00
$4.00
$512.00
$128.00
$32.00
$8.00
$2.00
$1,024.00
max{0, ($10.00 - $1024.00)} = $0.00
$256.00
max{0, ($10.00 - $256.00)} = $0.00
$64.00
max{0, ($10.00 - $64.00)} = $0.00
$16.00
max{0, ($10.00 - $16.00)} = $0.00
$4.00
max{0, ($10.00 - $4.00)} = $6.00
$1.00
max{0, ($10.00 - $1.00)} = $9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$1,024.00
$0.00
$512.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$256.00
$0.00
$128.00
$64.00
$32.00
$16.00
$8.00
$4.00
$2.00
$6.00
$1.00
$9.00
{ ($0 x 0.5000)
+ ($6.00 x 0.5000) }
/ exp(0.240227)
= $2.36
{ ($6.00 x 0.5000)
+ ($9.00 x 0.5000) }
/ exp(0.240227)
= $5.90

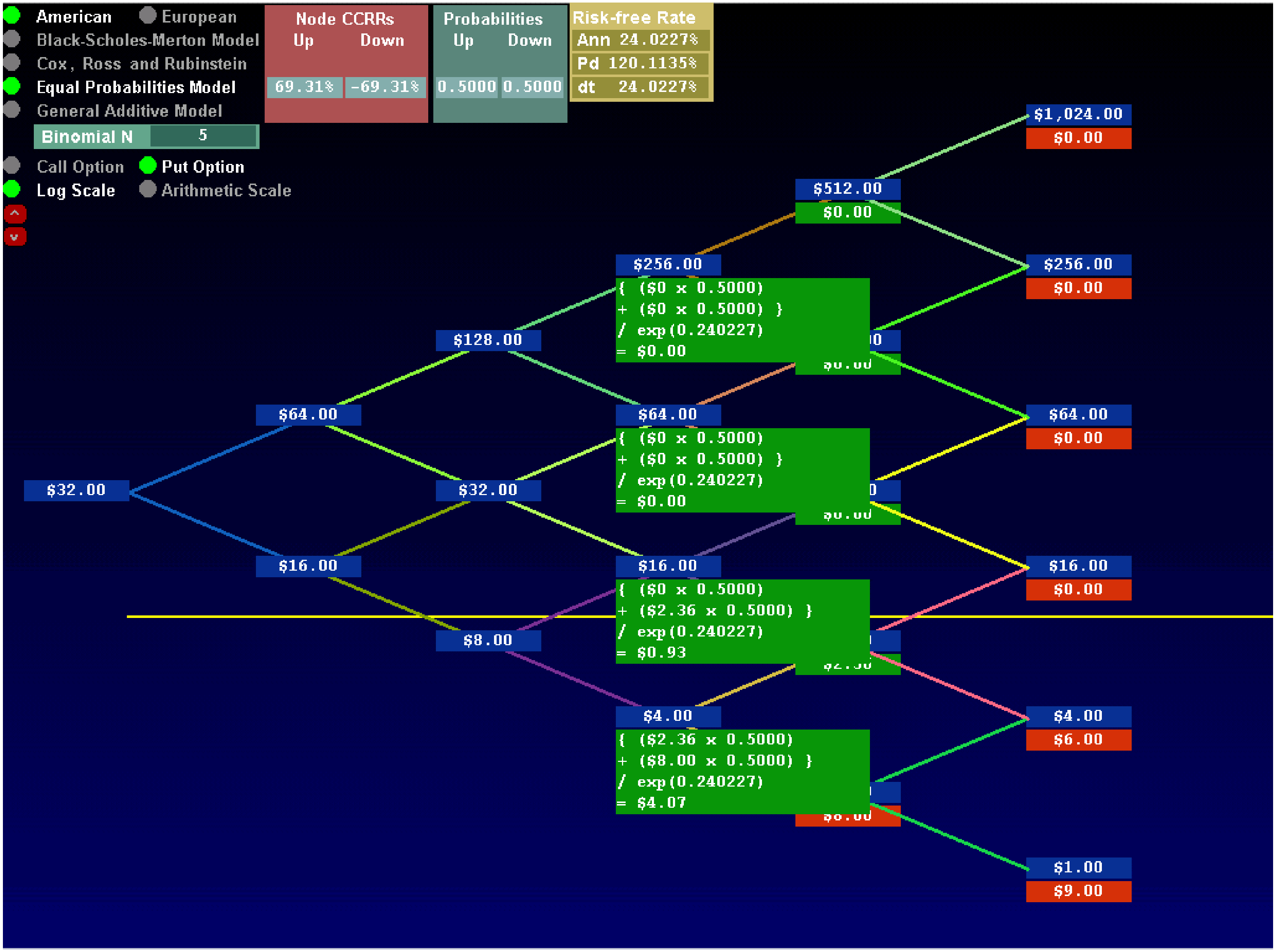
At this juncture, we add another step.

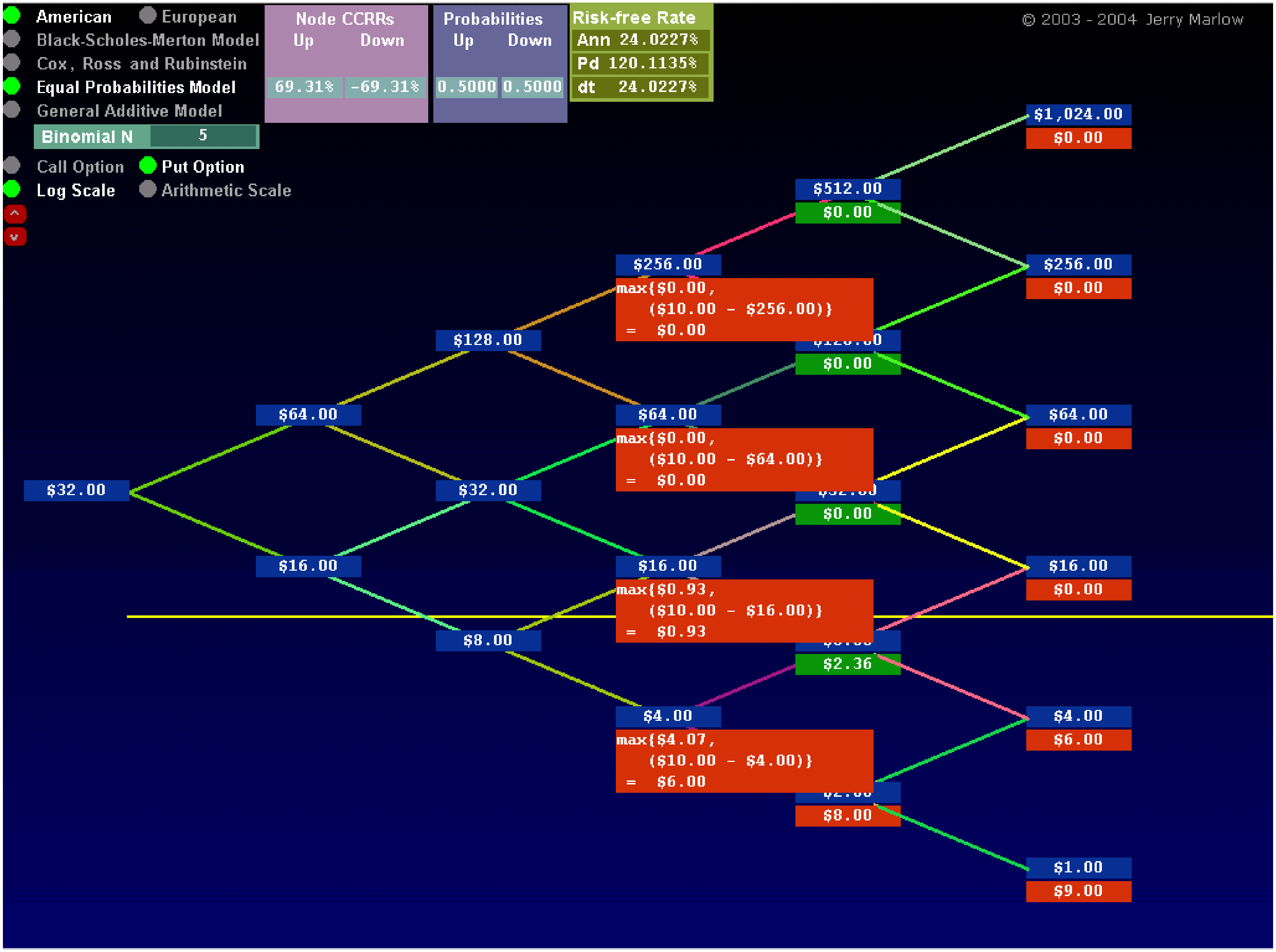
Recall that, in describing the binomial methodology, we said that, for each price path, we would calculate the maximum expected payoff of the option and the payoff's probability. At the $2.00-stock-price node, the point-in-time value of the subsequent payoffs is $5.90. Yet, if the stock price were $2.00 and the option owner exercised the option, then his or her payoff or point-in-time value would be $10.00 - $2.00 = $8.00. Accordingly, when valuing American-style options, at each node,

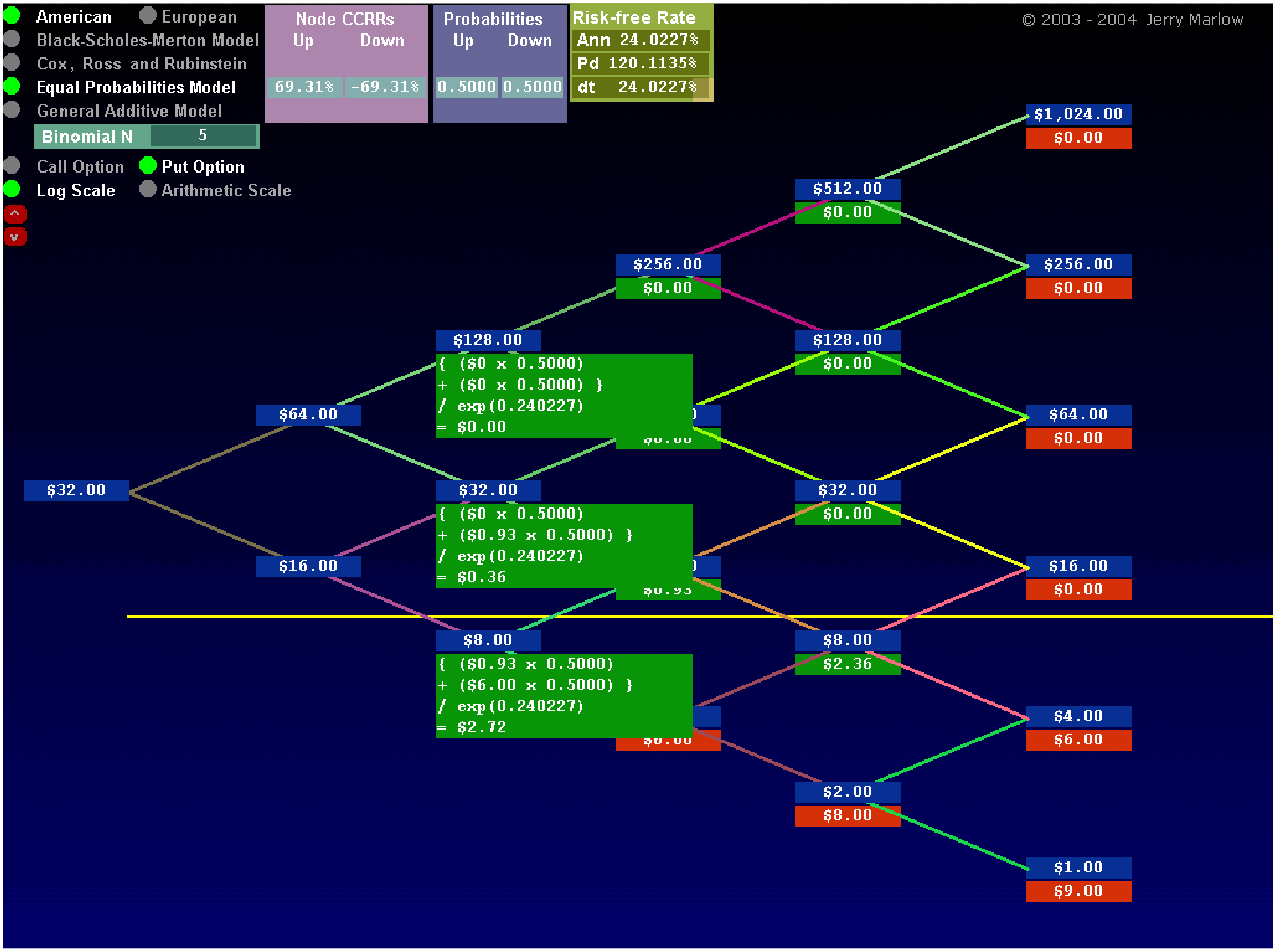

American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$32.00
$64.00
$16.00
$128.00
$32.00
$8.00
$256.00
$64.00
$16.00
$4.00
$512.00
max{$0.00, ($10.00 - $512.00)} = $0.00
$128.00
max{$0.00, ($10.00 - $128.00)} = $0.00
$32.00
max{$0.00, ($10.00 - $32.00)} = $0.00
$8.00
max{$2.36, ($10.00 - $8.00)} = $2.36
$2.00
max{$5.90, ($10.00 - $2.00)} = $8.00
$1,024.00
$0.00
$256.00
$0.00
$64.00
$0.00
$16.00
$0.00
$4.00
$6.00
$1.00
$9.00

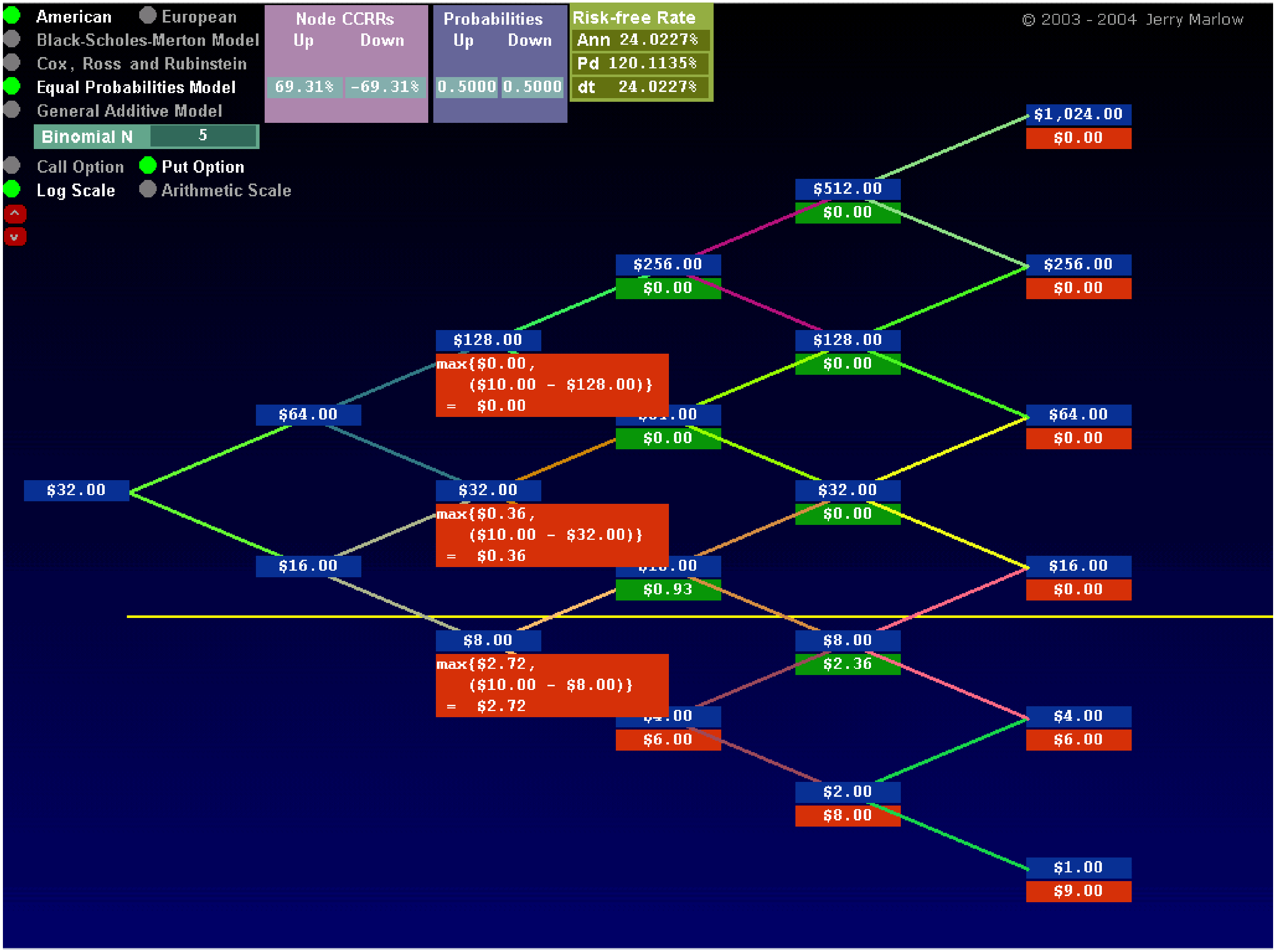
we compare the point-in-time value of the subsequent payoffs with the payoff at the present node. We retain the larger value and continue our backward induction with it.

Thus we proceed with valuation of American-style options by backward induction.


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
$32.00
$64.00
$16.00
$128.00
$32.00
$8.00
$256.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$64.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$16.00
{ ($0 x 0.5000)
+ ($2.36 x 0.5000) }
/ exp(0.240227)
= $0.93
$4.00
{ ($2.36 x 0.5000)
+ ($8.00 x 0.5000) }
/ exp(0.240227)
= $4.07
$512.00
$0.00
$1,024.00
$0.00
$256.00
$0.00
$64.00
$0.00
$16.00
$0.00
$4.00
$6.00
$1.00
$9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$1,024.00
$0.00
$512.00
$0.00
$256.00
max{$0.00,
($10.00 - $256.00)}
= $0.00
$256.00
$0.00
$128.00
$0.00
$64.00
$64.00
max{$0.00,
($10.00 - $64.00)}
= $0.00
$64.00
$0.00
$32.00
$32.00
$0.00
$16.00
$16.00
max{$0.93,
($10.00 - $16.00)}
= $0.93
$16.00
$0.00
$8.00
$2.36
$4.00
max{$4.07,
($10.00 - $4.00)}
= $6.00
$4.00
$6.00
$8.00
$1.00
$9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$1,024.00
$0.00
$512.00
$0.00
$256.00
$0.00
$256.00
$0.00
$128.00
{ ($0 x 0.5000)
+ ($0 x 0.5000) }
/ exp(0.240227)
= $0.00
$128.00
$0.00
$64.00
$64.00
$0.00
$32.00
$32.00
{ ($0 x 0.5000)
+ ($0.93 x 0.5000) }
/ exp(0.240227)
= $0.36
$32.00
$0.00
$16.00
$16.00
$0.00
$8.00
{ ($0.93 x 0.5000)
+ ($6.00 x 0.5000) }
/ exp(0.240227)
= $2.72
$8.00
$2.36
$4.00
$6.00
$2.00
$8.00
$1.00
$9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$1,024.00
$0.00
$512.00
$0.00
$256.00
$0.00
$256.00
$0.00
$128.00
max{$0.00,
($10.00 - $128.00)}
= $0.00
$128.00
$0.00
$64.00
$0.00
$64.00
$0.00
$32.00
$32.00
max{$0.36,
($10.00 - $32.00)}
= $0.36
$32.00
$0.00
$16.00
$0.93
$16.00
$0.00
$8.00
max{$2.72,
($10.00 - $8.00)}
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$2.36
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$4.00
$6.00
$2.00
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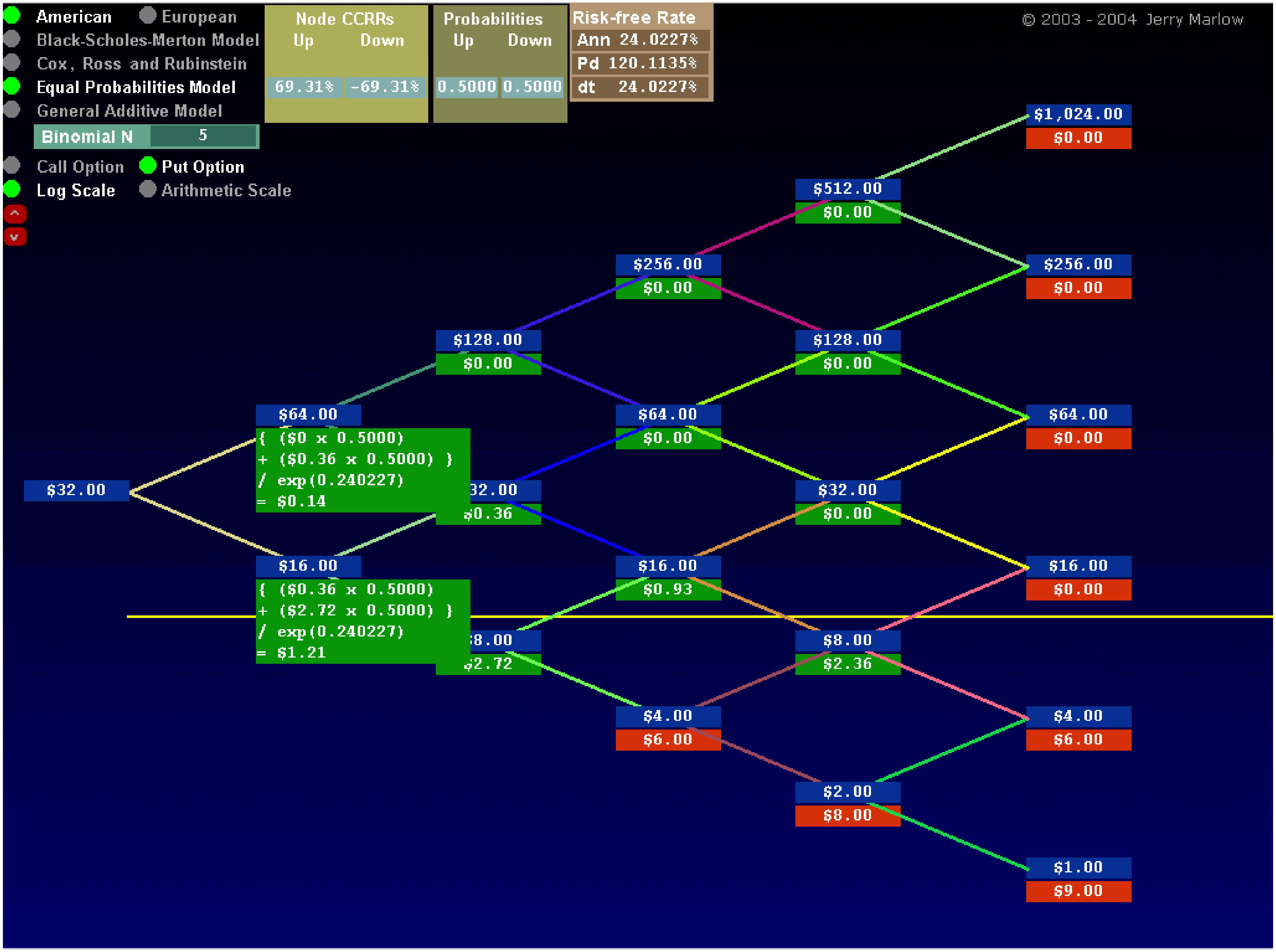
$1.00
$9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$32.00
$64.00
{ ($0 x 0.5000)
+ ($0.36 x 0.5000) }
/ exp(0.240227)
= $0.14
$16.00
{ ($0.36 x 0.5000)
+ ($2.72 x 0.5000) }
/ exp(0.240227)
= $1.21
$128.00
$0.00
32.00
$0.36
8.00
$2.72
$256.00
$0.00
$64.00
$0.00
$16.00
$0.93
$4.00
$6.00
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$8.00
$2.36
$2.00
$8.00
$1,024.00
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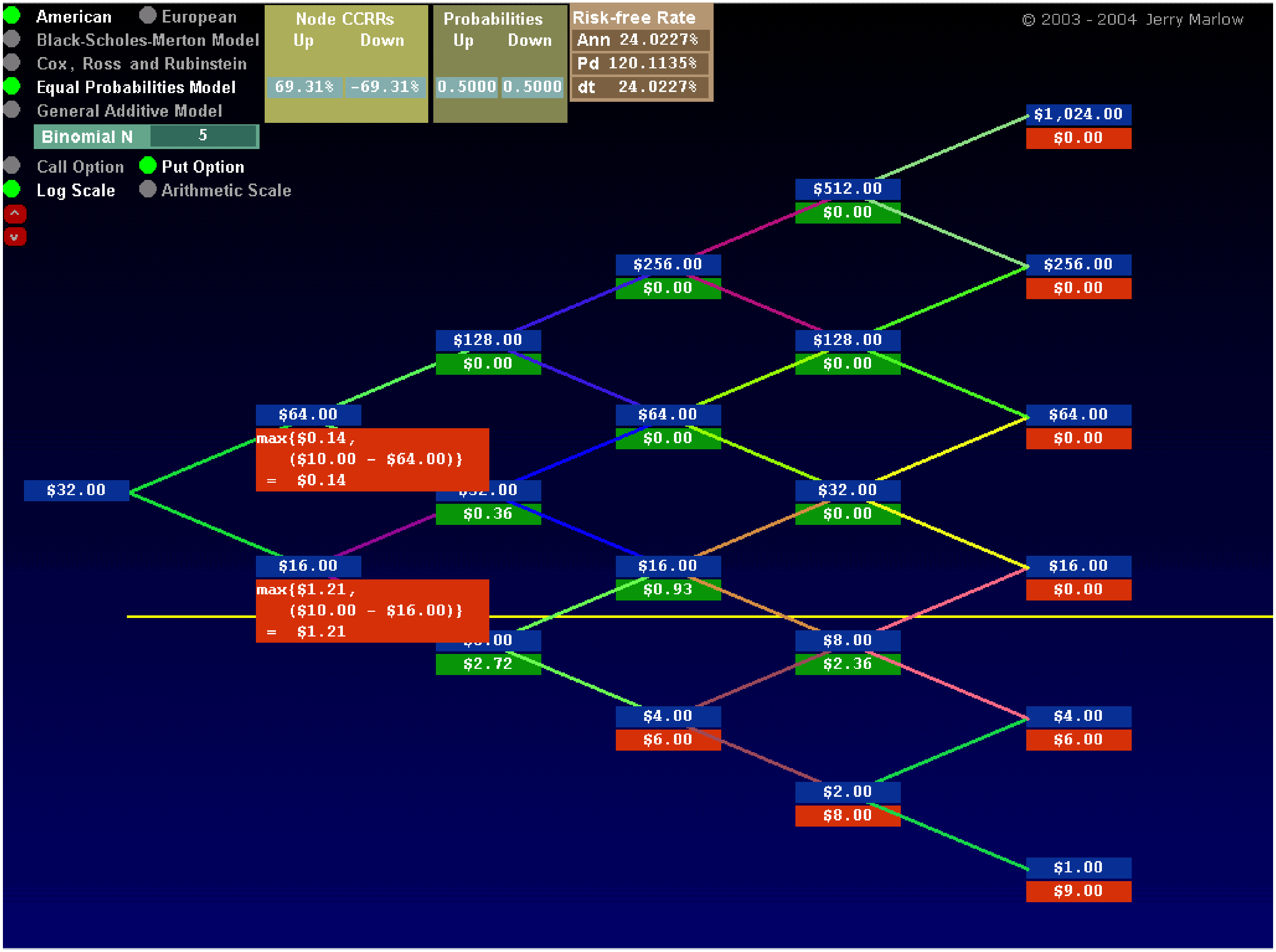
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European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$32.00
$64.00
max{$0.14,
($10.00 – $64.00)}
= $0.14
$16.00
max{$1.21,
($10.00 – $16.00)}
= $1.21
$128.00
$0.00
$0.36
$2.72
$256.00
$0.00
$64.00
$0.00
$16.00
$0.93
$4.00
$6.00
$512.00
$0.00
$128.00
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$32.00
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$8.00
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$2.00
$8.00
$1,024.00
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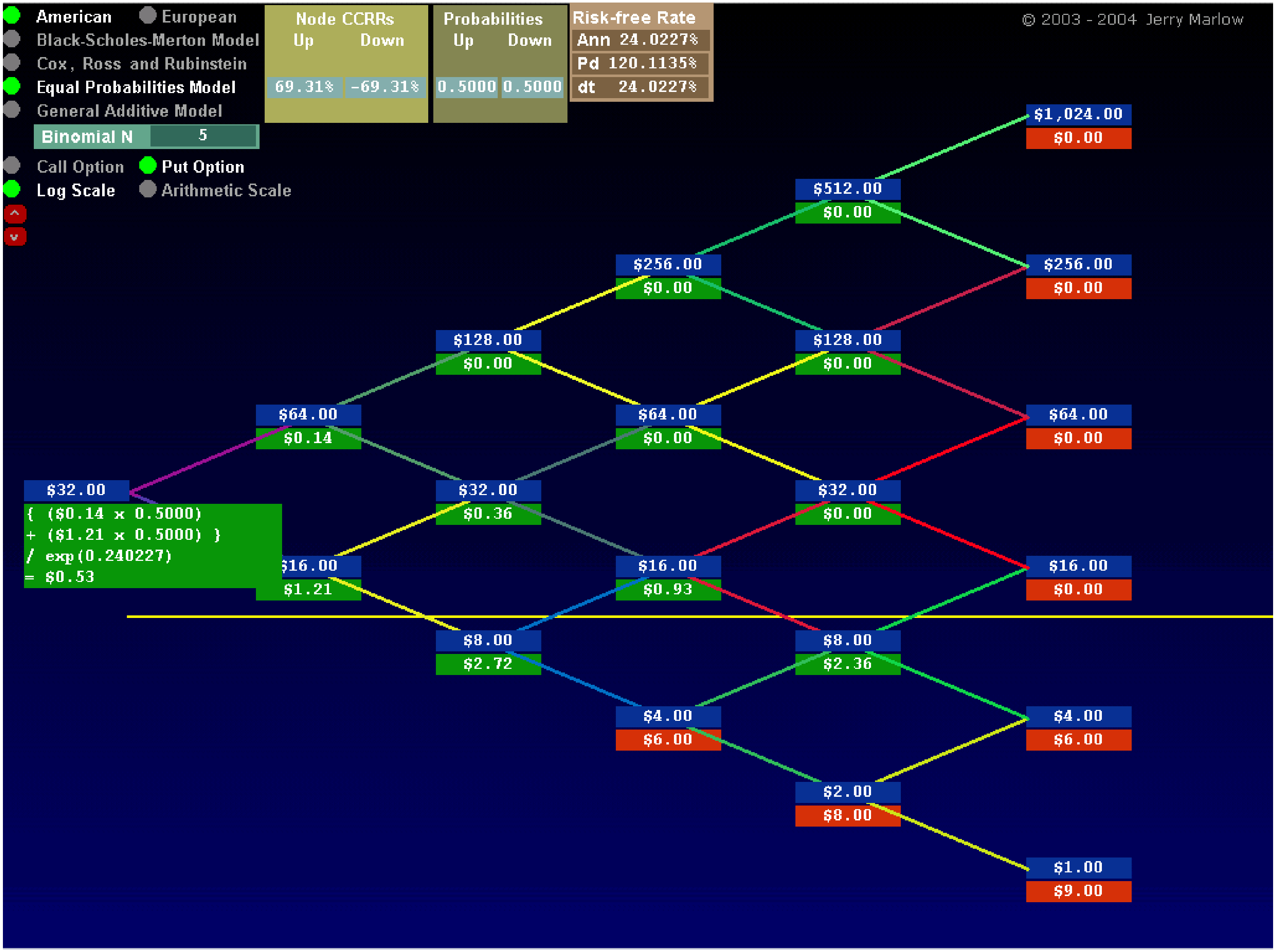
$9.00


American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
$1,024.00
$0.00
$512.00
$0.00
$256.00
$0.00
$256.00
$0.00
$128.00
$0.00
$128.00
$0.00
$64.00
$0.14
$64.00
$0.00
$64.00
$0.00
$32.00
{ ($0.14 x 0.5000)
+ ($1.21 x 0.5000) }
/ exp(0.240227)
= $0.53
$32.00
$0.36
$32.00
$0.00
$16.00
$1.21
$16.00
$0.93
$16.00
$0.00
$8.00
$2.72
$8.00
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$2.00
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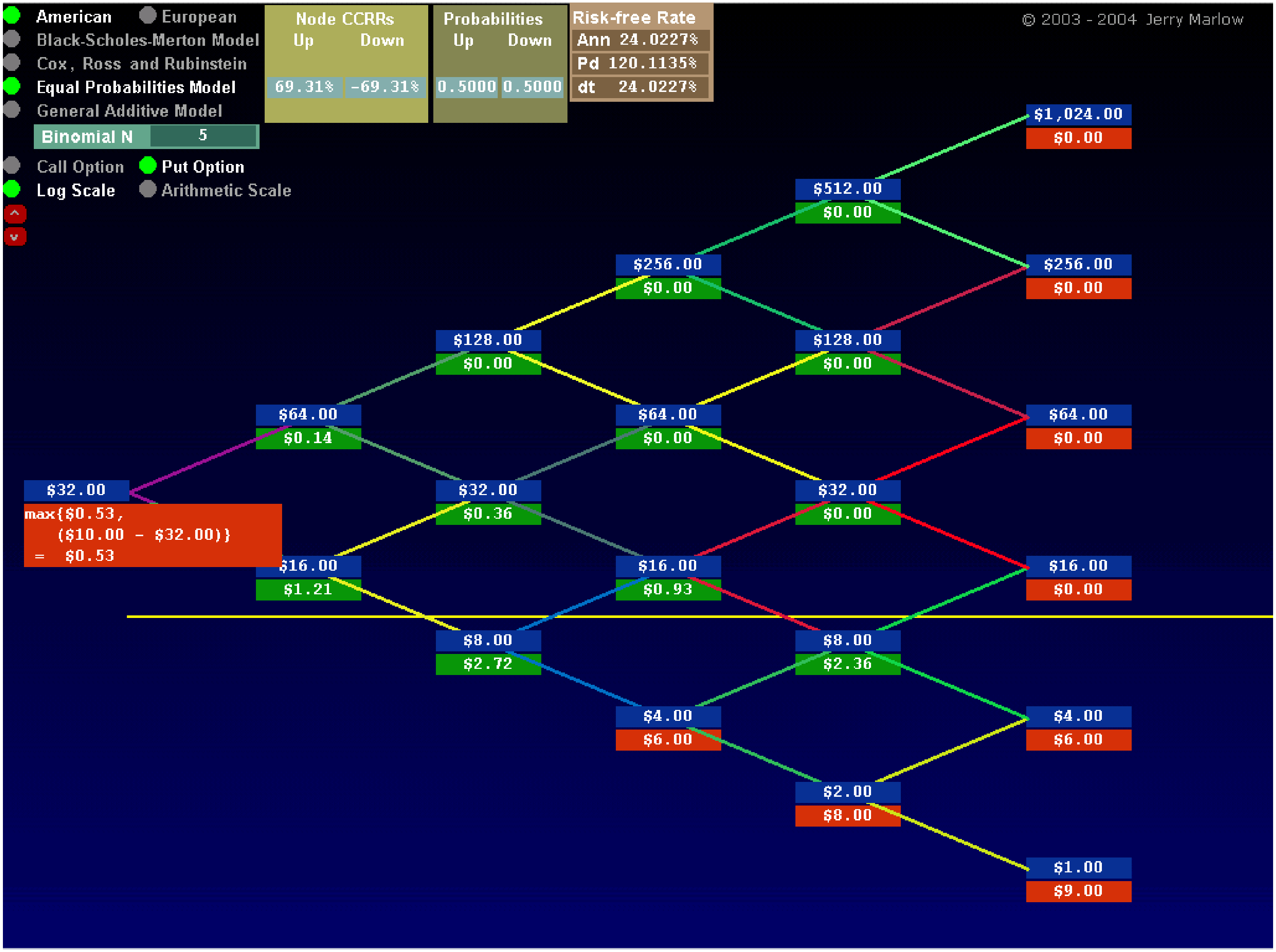
$9.00


American
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Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 5
Node CCRRs
Up Down
69.31% -69.31%
Probabilities
Up Down
0.5000 0.5000
Risk-free Rate
Ann 24.0227%
Pd 120.1135%
dt 24.0227%
© 2003 - 2004 Jerry Marlow
Call Option
Put Option
Log Scale
Arithmetic Scale
$1,024.00
$0.00
$512.00
$0.00
$256.00
$0.00
$256.00
$0.00
$128.00
$0.00
$128.00
$0.00
$64.00
$0.14
$64.00
$0.00
$64.00
$0.00
$32.00
max{$0.53,
($10.00 - $32.00)}
= $0.53
$32.00
$0.36
$32.00
$0.00
$16.00
$1.21
$16.00
$0.93
$16.00
$0.00
$8.00
$2.72
$8.00
$2.36
$4.00
$6.00
$4.00
$6.00
$2.00
$8.00
$1.00
$9.00

For this American-style put option, including the potential value of early exercise gives an option value of $0.53 versus the value of $0.37 for the otherwise identical European-style option.

Given the up and down continuously compounded rates of return and the up and down probabilities, valuing European- and American-style options (at least on underlyings that pay no dividends) is reasonably straightforward. But how does one come up with the up and down rates of return and probabilities?

Binomial models translate forecasts of expected return and volatility into up and down rates of return and probabilities

In essence, binomial models translate the geometric-Brownian-motion framework into the binomial framework. When using risk-neutral valuation models, we assume that the stock's expected return is equal to the risk-free rate. The uncertainty of the forecast is expressed as the standard deviation of the stock price's expected volatility. Binomial models translate this expected return and the underlying stock's expected volatility into up and down continuously compounded rates of return and into up and down probabilities.

Different binomial models make the translations differently. Here we look at how three different binomial models: an Equal-Probabilities Model; the Cox, Ross, and Rubinstein model; and the General Additive Model; make the translation.

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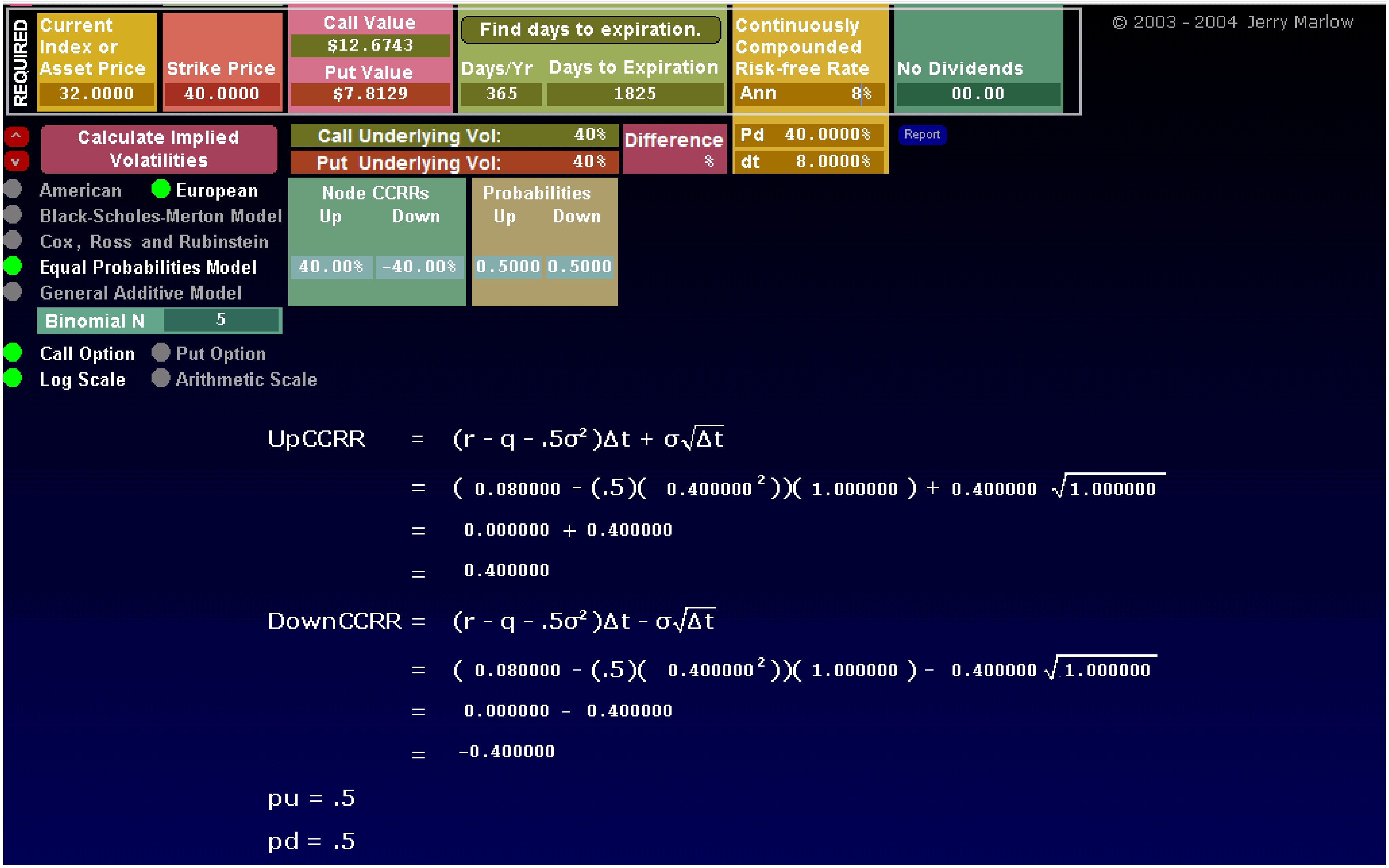


The Equal Probabilities Model sets the up probability and the down probability equal to .5.

Recall that, in our discussion of the geometric-Brownian-motion model, we said that a stock forecast's middle or median return is equal to the expected return minus half the standard deviation squared. If a stock pays a dividend yield q then, when calculating the median return, the dividend yield is subtracted from the expected return.

In the Equal Probabilities Model, the up continuously compounded rate of return for each time step is set equal to the median return for the time step plus the standard deviation of volatility for the time step. The down continuously compounded rate of return for each time step is set equal to the median return for the time step minus the standard deviation of volatility for the time step.

The up probability is set equal to .5. So is the down probability.

In the example above, we use a risk-free rate of 8% and an expected volatility of 40% which gives a median return of 0%.

The investment horizon is five years with five time steps. Hence, for each time step, Δt is equal to one.

Under these conditions, the model gives up and down continuously compounded rates of return of 40% and -40%.

$$\text{UpCCRR} = (r - q - .5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}$$

$$= (\ 0.130000 - (.5)(\ 0.400000^2))(\ 1.000000\ ) + 0.400000\ \sqrt{1.000000}$$

$$= 0.050000 + 0.400000$$

$$= 0.450000$$

$$\text{DownCCRR} = (r - q - .5\sigma^2)\Delta t - \sigma\sqrt{\Delta t}$$

$$= (\ 0.130000 - (.5)(\ 0.400000^2))(\ 1.000000\ ) - 0.400000\ \sqrt{1.000000}$$

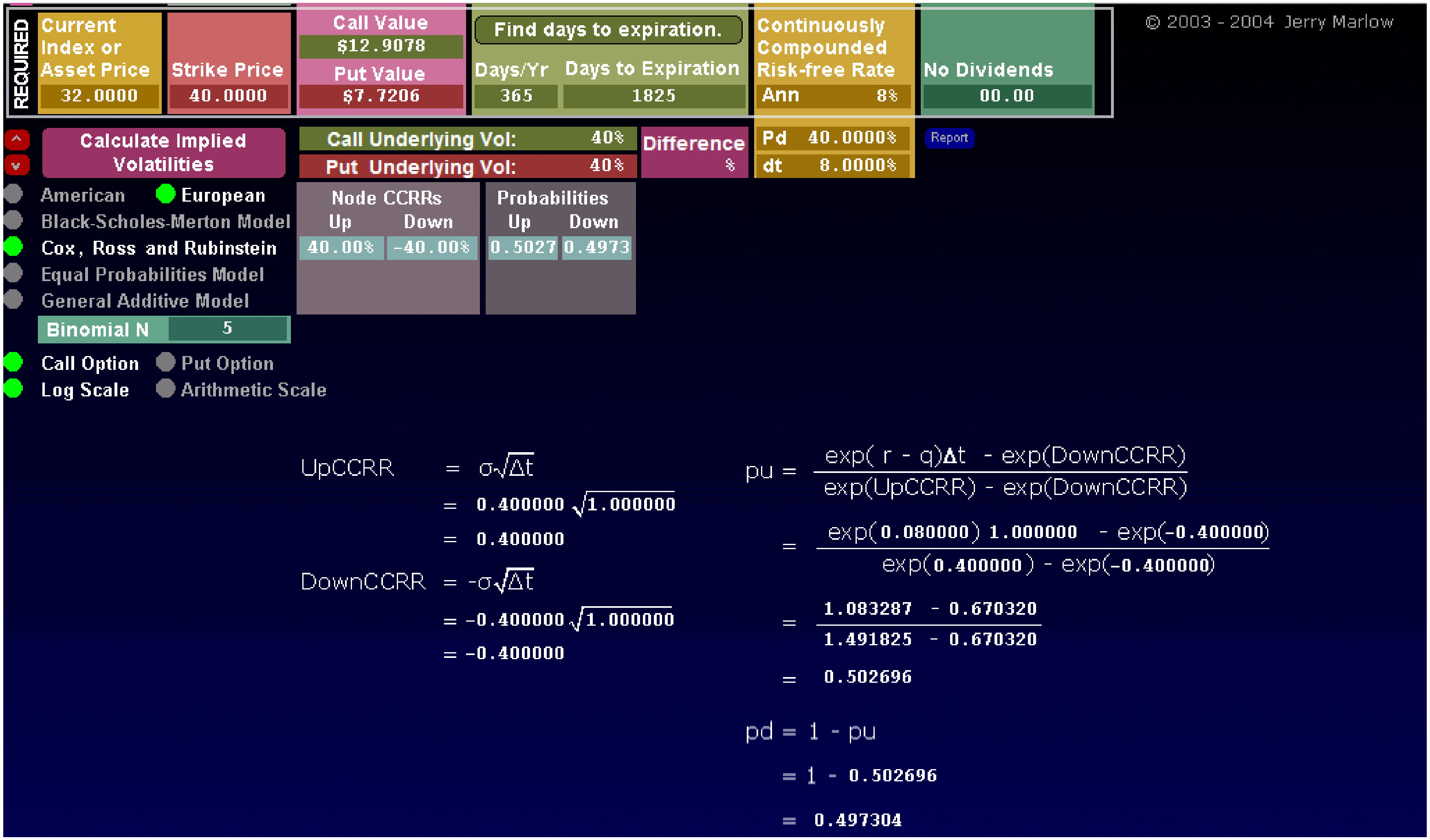
$$= 0.050000 - 0.400000$$

$$= -0.350000$$

pu = .5

pd = .5

If we set r, the expected return, equal to 13%, then the time-step median return is 5%; the up continuously compounded rate of return for the time step is 45%; the down continuously compounded rate of return for the time step is -35%; the up and down probabilities are .5.



The Cox, Ross, and Rubinstein Model makes the up and down jumps equal.

Recall that volatility varies with the square root of time. The Cox, Ross, and Rubinstein Model sets the up continuously compounded rate of return for the time step equal to the stock price's expected volatility for the time interval.

It sets the down continuously compounded rate of return equal to minus one times the stock price's expected volatility for the time interval.

The model adjusts the up probability to account for the stock price's positive expected return.

The down probability is one minus the up probability.

Given an expected return of 8%, a standard deviation of 40%, a five-year investment horizon and five times steps, the up continuously compounded rate of return is 40%. The down continuously compounded rate of return is -40%. The up probability is 0.502696. The down probability is 0.497304.

$$\text{UpCCRR} = \sigma\sqrt{\Delta t} = 0.400000\sqrt{1.000000} = 0.400000$$

$$\text{DownCCRR} = -\sigma\sqrt{\Delta t} = -0.400000\sqrt{1.000000} = -0.400000$$

$$pu = \frac{\exp(r - q)\Delta t - \exp(\text{DownCCRR})}{\exp(\text{UpCCRR}) - \exp(\text{DownCCRR})}$$

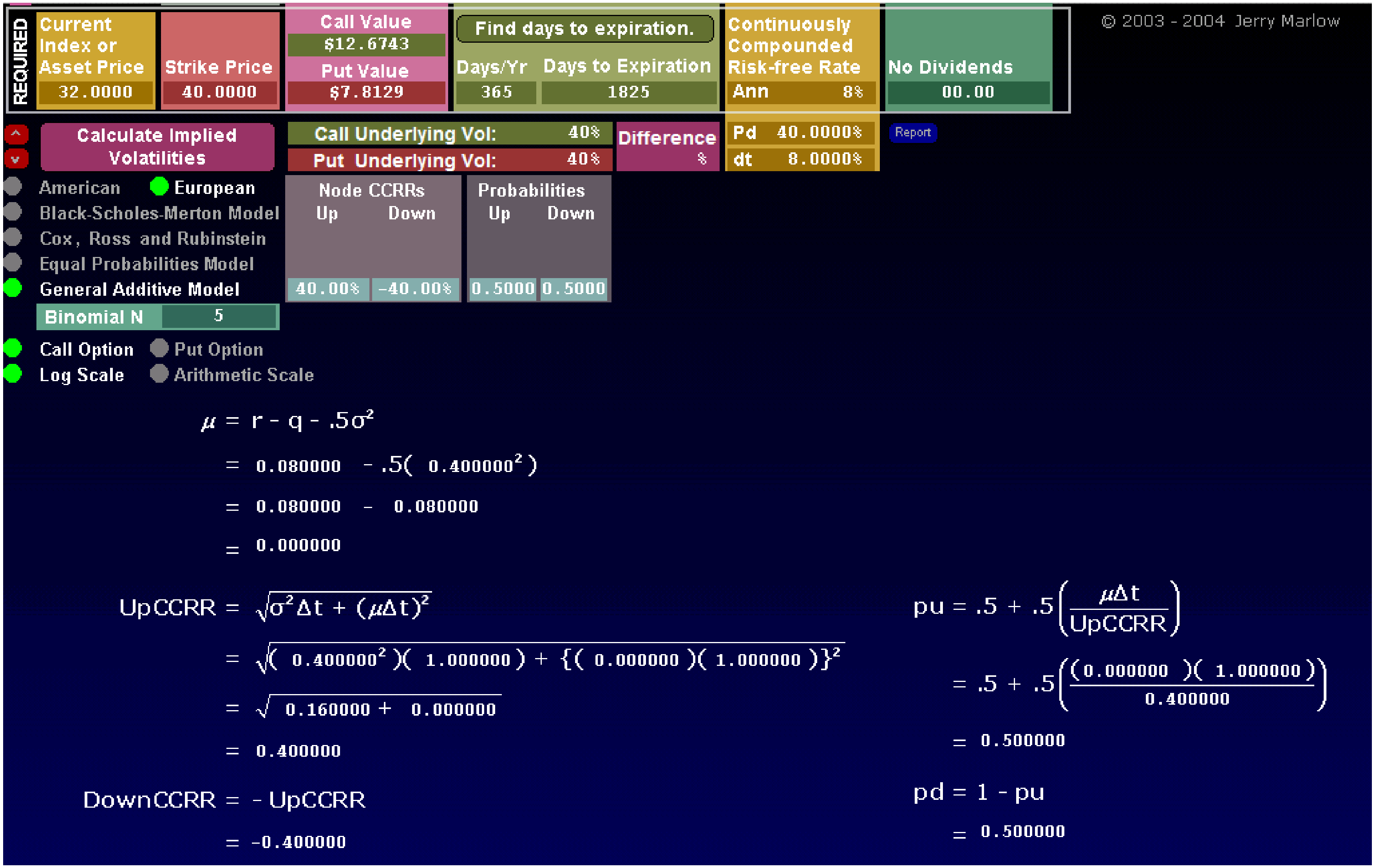
$$= \frac{\exp(0.130000)\,1.000000 - \exp(-0.400000)}{\exp(0.400000) - \exp(-0.400000)}$$

$$= \frac{1.138828 - 0.670320}{1.491825 - 0.670320}$$

$$= 0.570305$$

$$pd = 1 - pu = 1 - 0.570305 = 0.429695$$

If we change the stock price’s expected return to 13%, the up and down continuously compounded rates of return remain at 40% and -40% respectively. The up probability becomes 0.570305. The down probability becomes 0.429695.



The General Additive Model makes the up and down jumps equal.

The General Additive Model first computes mu and then uses mu to compute the up and down continuously compounded rates of return and probabilities.

Given an expected return of 8%, a standard deviation of 40%, a five-year investment horizon and five times steps, mu is equal to zero. The up and down continuously compounded rates of return are 40% and -40%. The up and down probabilities are 0.5.

$$\mu = r - q - .5\sigma^2$$
$$= 0.130000 - .5(0.400000^2)$$
$$= 0.130000 - 0.080000$$
$$= 0.050000$$

$$UpCCRR = \sqrt{\sigma^2 \Delta t + (\mu \Delta t)^2}$$
$$= \sqrt{(0.400000^2)(1.000000) + \{(0.050000)(1.000000)\}^2}$$
$$= \sqrt{0.160000 + 0.002500}$$
$$= 0.403113$$

$$DownCCRR = -UpCCRR$$
$$= -0.403113$$

$$pu = .5 + .5\left(\frac{\mu \Delta t}{UpCCRR}\right)$$
$$= .5 + .5\left(\frac{(0.050000)(1.000000)}{0.403113}\right)$$
$$= 0.562017$$

$$pd = 1 - pu$$
$$= 0.437983$$

If we change the expected return to 13%, the model gives a mu of 5%. Up and down continuously compounded rates of return are 40.3113% and -40.3113%. The up probability is 0.562017 and the down probability is 0.437983.

Where do these formulas come from?

Binomial models model the arbitrage-free evolution of stock prices. Arbitrage free evolution means that, at the end of each time interval, the probability-weighted average of stock prices in that stack must offer a return over the initial stock price equal to the risk-free rate. Looking at the evolution over just one time interval, we can see how the Cox, Ross, and Rubinstein Model calculates its up and down probabilities.

We start by setting the condition that the probability-weighted result of the up and down movements of the initial stock price, S_0, must equal the initial stock price growing at the risk-free rate. (To simplify, we leave out any dividend yield, q.)

$$pu \cdot S_0 \cdot \exp(UpCCRR) + (1 - pu) \cdot S_0 \cdot \exp(DownCCRR) = S_0 \cdot \exp(r\Delta t)$$

To solve for pu, we manipulate and rearrange our equation:

$$pu \cdot S_0 \cdot \exp(UpCCRR) + S_0 \cdot \exp(DownCCRR) - pu \cdot S_0 \cdot \exp(DownCCRR) = S_0 \cdot \exp(r\Delta t)$$

$$pu\{S_0 \cdot \exp(UpCCRR) - S_0 \cdot \exp(DownCCRR)\} = S_0 \cdot \exp(r\Delta t) - S_0 \cdot \exp(DownCCRR)$$

$$pu \cdot S_0\{\exp(UpCCRR) - \exp(DownCCRR)\} = S_0\{\exp(r\Delta t) - \exp(DownCCRR)\}$$

$$\frac{pu \cdot S_0\{\exp(UpCCRR) - \exp(DownCCRR)\}}{S_0} = \frac{S_0\{\exp(r\Delta t) - \exp(DownCCRR)\}}{S_0}$$

$$pu\{\exp(UpCCRR) - \exp(DownCCRR)\} = \exp(r\Delta t) - \exp(DownCCRR)$$

$$pu = \frac{\exp(r\Delta t) - \exp(DownCCRR)}{\exp(UpCCRR) - \exp(DownCCRR)}$$

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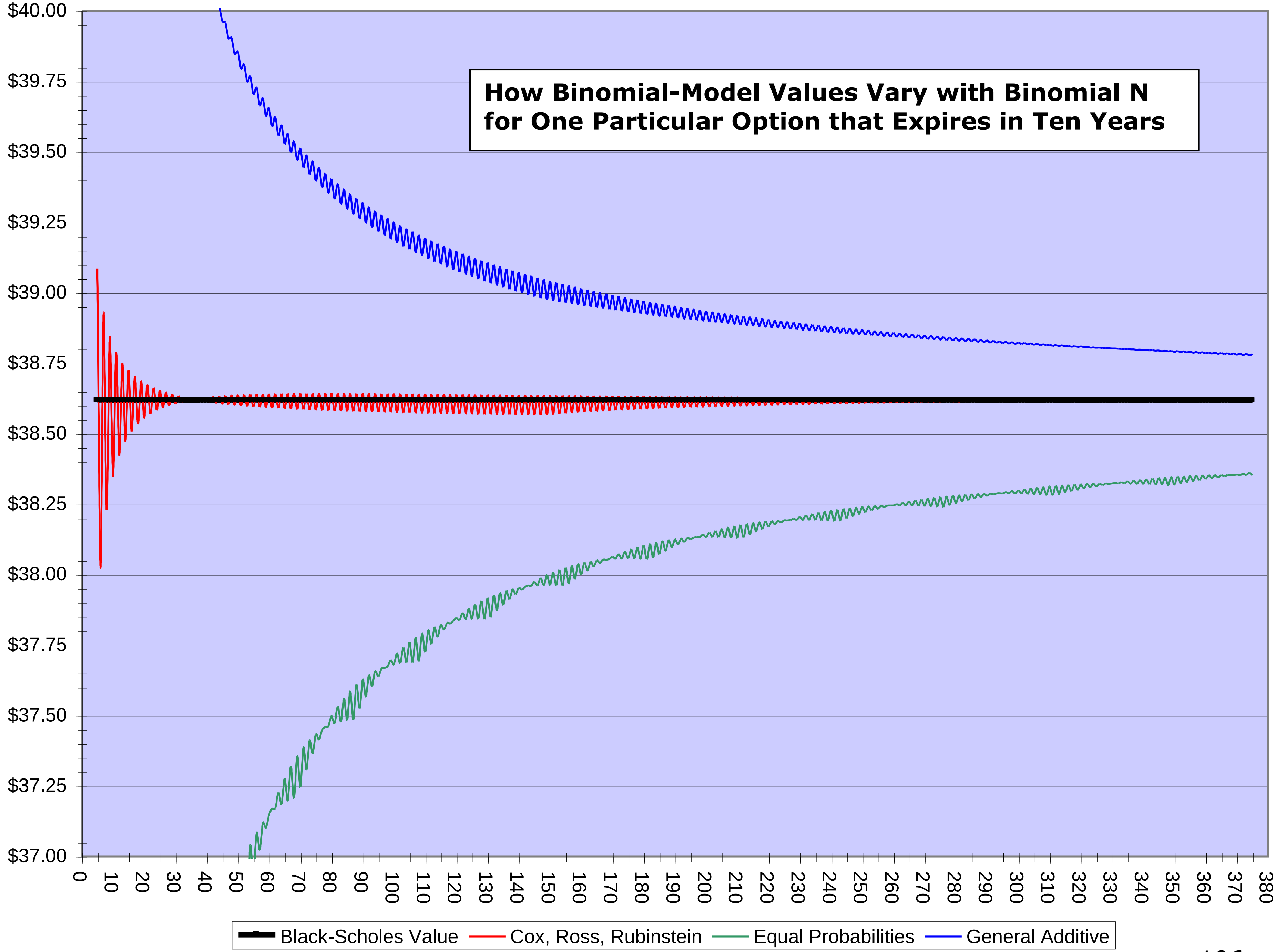

How Binomial-Model Values Vary with Binomial N
for One Particular Option that Expires in Ten Years
$40.00
$39.75
$39.50
$39.25
$39.00
$38.75
$38.50
$38.25
$38.00
$37.75
$37.50
$37.25
$37.00
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250 260 270 280 290 300 310 320 330 340 350 360 370 380
Black-Scholes Value
Cox, Ross, Rubinstein
Equal Probabilities
General Additive

In general, the translation of the geometric-Brownian-motion model into the binomial model is more accurate as the option's time to expiration is divided into shorter time intervals; but beware!

When we looked at the geometric-Brownian-motion model, we saw that when we divided the bell-shaped curve into a sufficiently large number of possible outcomes and set the stock's expected return equal to the risk-free rate, then the option value we arrived at was equal to the option's Black-Scholes value. We also noted that, for call options whose underlyings pay no dividends, there is no advantage to early exercise. Hence, for these options, the values of a European-style and an American-style option are the same.

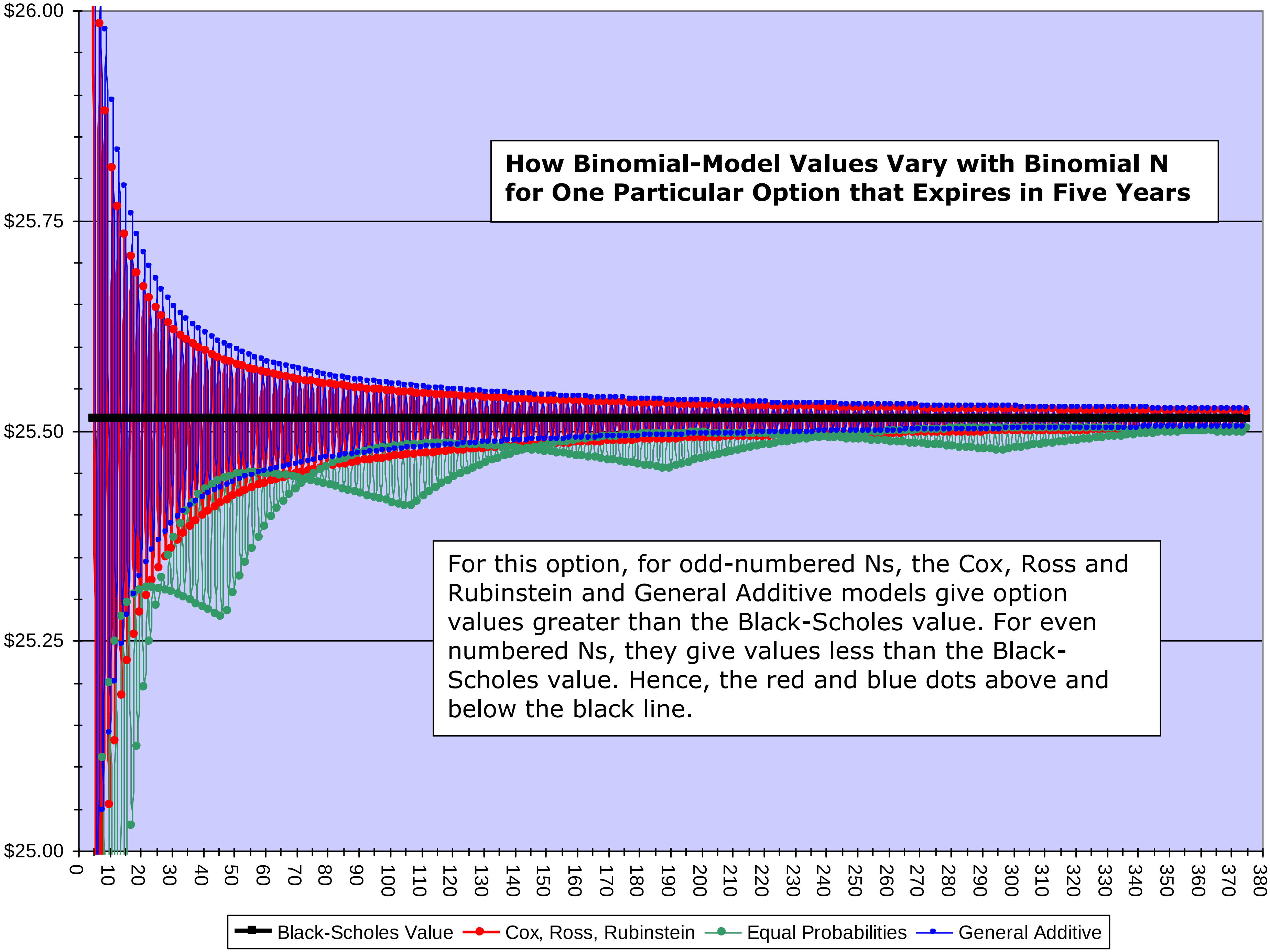
To see how well a binomial model translates geometric-Brownian-motion into a binomial evolution of stock prices, we can compare, for call options whose underlyings pay no dividends, the option values that the binomial models compute with the option's Black-Scholes value.

Above we value an American-style call option on an underlying that pays no dividends. We look at how changing binomial N changes how close the binomial models' values approach the Black-Scholes value. The underlying stock's current asset price is $50.00. The strike price is $60.00. The risk-free rate is 5%. The stock's expected volatility is 70%. Time to expiration is 10 years. The option's Black-Scholes value is $38.62.

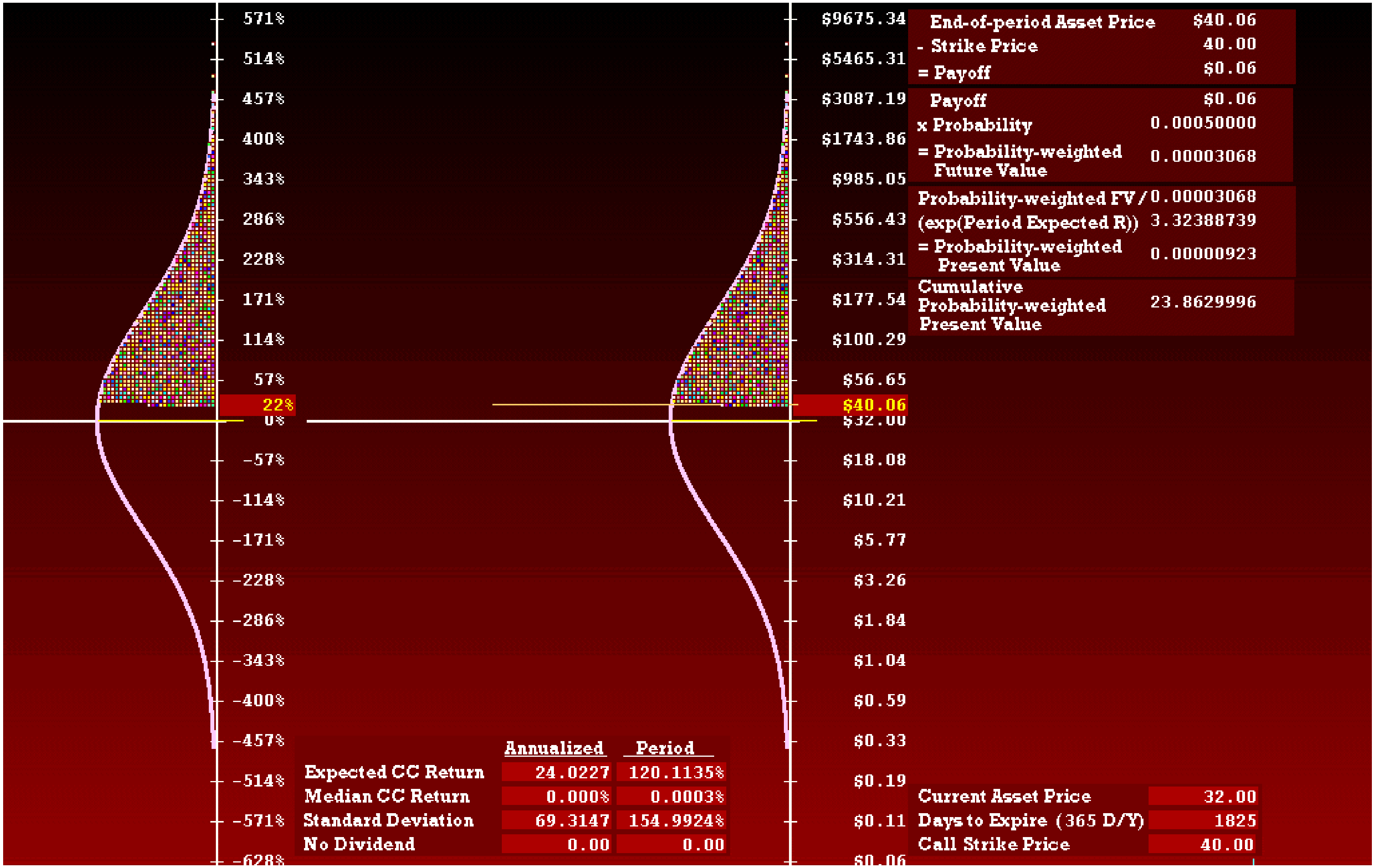
In general, the greater binomial N, the shorter the time interval of each step, and the closer the binomial values approach the Black-Scholes value. However, the convergence with the Black-Scholes value varies by model and by option. Even- and odd-numbered binomial Ns give noticeably different values; hence the squiggles in the value lines. For this option, at a binomial N of 34, the Cox, Ross and Rubinstein model gives an option value very close to the Black-Scholes value but then gives more divergent values until binomial N reaches 322.

Option-pricing models were developed to price market-traded options, most of which have times to expiration of less than one year. Employee stock options when issued, by contrast, frequently have times of expiration of up to ten years. When choosing a binomial model with which to value employee stock options, you may wish to compare the option values that candidate models give with the value that the Black-Scholes model gives. In making this comparison, you may wish to see how changing binomial N affects the proximity of the values the different models produce.

To give an idea of how the convergence of binomial values can vary by option, below we look at the pattern of values for a call option that expires in 5 years. The current stock price is $50.00. The strike price is $50.00. The risk-free rate is 6%. The stock's expected volatility is 50%.

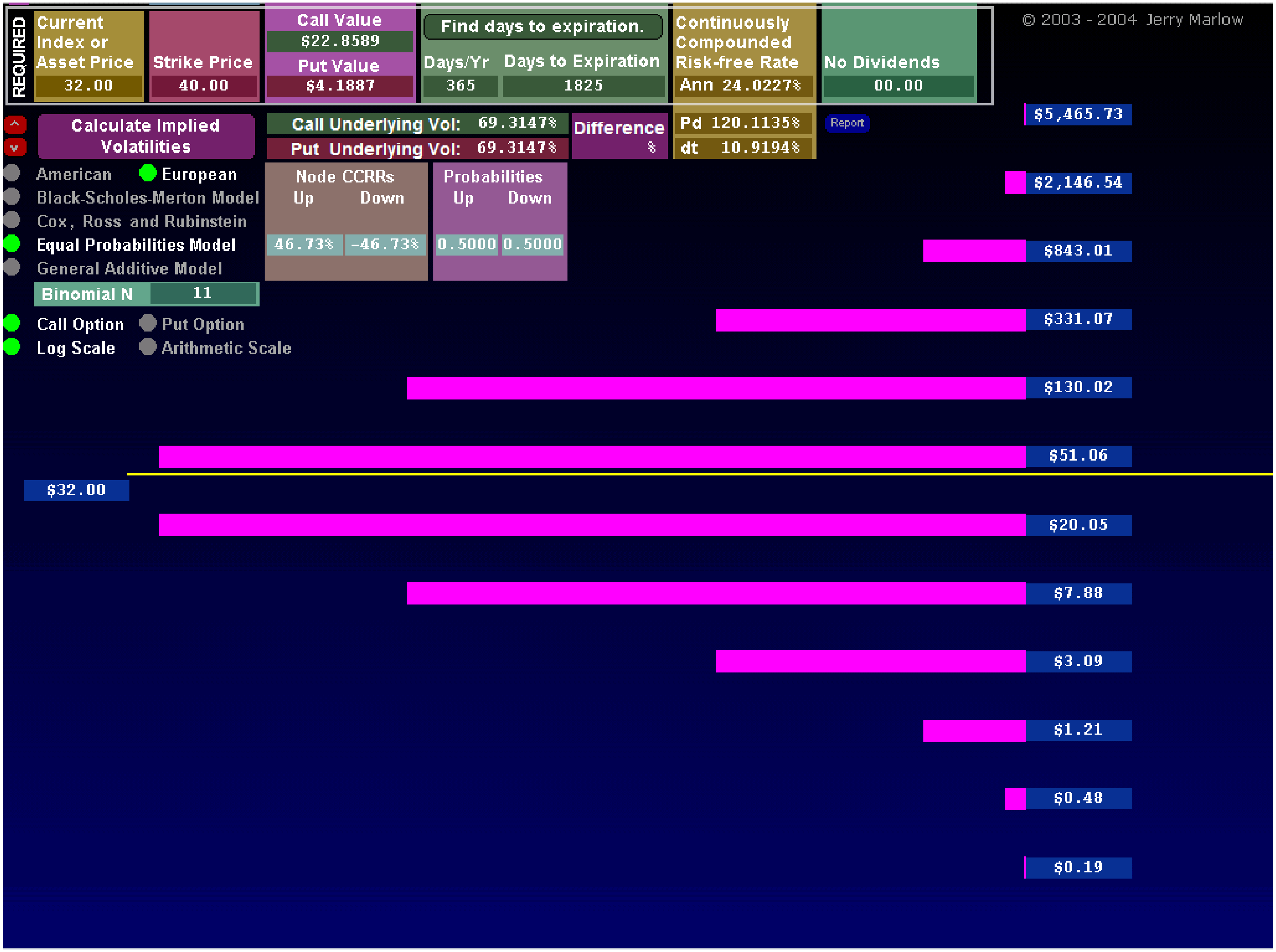


$26.00
$25.75
$25.50
$25.25
$25.00
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250 260 270 280 290 300 310 320 330 340 350 360 370 380
How Binomial-Model Values Vary with Binomial N for One Particular Option that Expires in Five Years
For this option, for odd-numbered Ns, the Cox, Ross and Rubinstein and General Additive models give option values greater than the Black-Scholes value. For even numbered Ns, they give values less than the Black-Scholes value. Hence, the red and blue dots above and below the black line.
Black-Scholes Value
Cox, Ross, Rubinstein
Equal Probabilities
General Additive



Risk-neutral models evaluate probability distributions against strike prices.

Conceptually speaking, when we use risk-neutral valuation to value European-style options and American-style call options on underlyings that pay no dividends, whether we use the geometric-Brownian-motion model or a binomial model, we are valuing a probability distribution against a strike price. In the geometric-Brownian-motion model we've been using, we arbitrarily divided the bell-shaped curve into 2,000 possible stock-price outcomes.

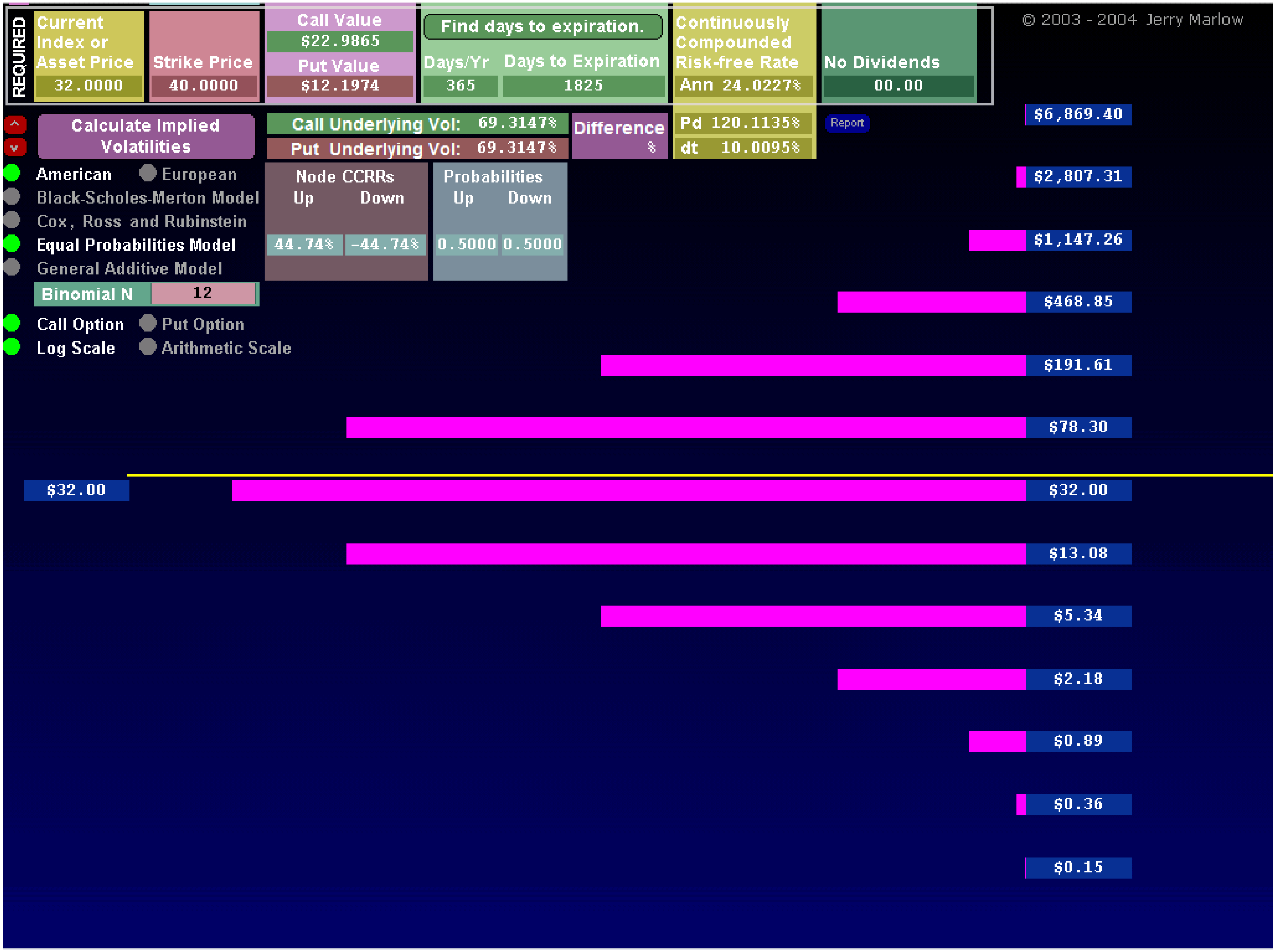

REQUIRED
Current Index or Asset Price
32.00
Strike Price
40.00
Call Value
$22.8589
Put Value
$4.1887
Find days to expiration.
Days/Yr
365
Days to Expiration
1825
Continuously Compounded Risk-free Rate
Ann 24.0227%
No Dividends
00.00
© 2003 - 2004 Jerry Marlow
Calculate Implied Volatilities
Call Underlying Vol: 69.3147%
Put Underlying Vol: 69.3147%
Difference
%
Pd 120.1135%
dt 10.9194%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 11
Node CCRRs
Up Down
46.73% -46.73%
Probabilities
Up Down
0.5000 0.5000
Call Option
Put Option
Log Scale
Arithmetic Scale
$5,465.73
$2,146.54
$843.01
$331.07
$130.02
$51.06
$32.00
$20.05
$7.88
$3.09
$1.21
$0.48
$0.19

If we set a binomial model's number of time steps to eleven, then the model generates a binomial probability distribution of 2,048 possible stock-price outcomes. Visually we can compare evaluating a strike price against a bell-shaped curve with evaluating it against a binomial distribution.

Using the geometric-Brownian-motion model, our probability distribution of end-of-investment-horizon prices is a bell-shaped curve. Bell-shaped curves are continuous. When carried to a sufficient number of decimal places, no end-of-horizon price occurs more than once. Accordingly, end-of-horizon stock prices do not bunch up immediately above, below, or at strike prices.

In the binomial model, by contrast, with eleven time steps (above), we have only twelve possible outcomes. In this example, with an initial stock price of $32.00 and a strike price of $40.00, using the Equal Probabilities Model, the highest stock-price outcome is $5,465.73. Half the terminal stock-price outcomes are above the strike price; half are below.

If we increase binomial N from 11 to 12, then the highest stock-price outcome is $6,869.40, but we now have significantly fewer than half the outcomes above the strike price.

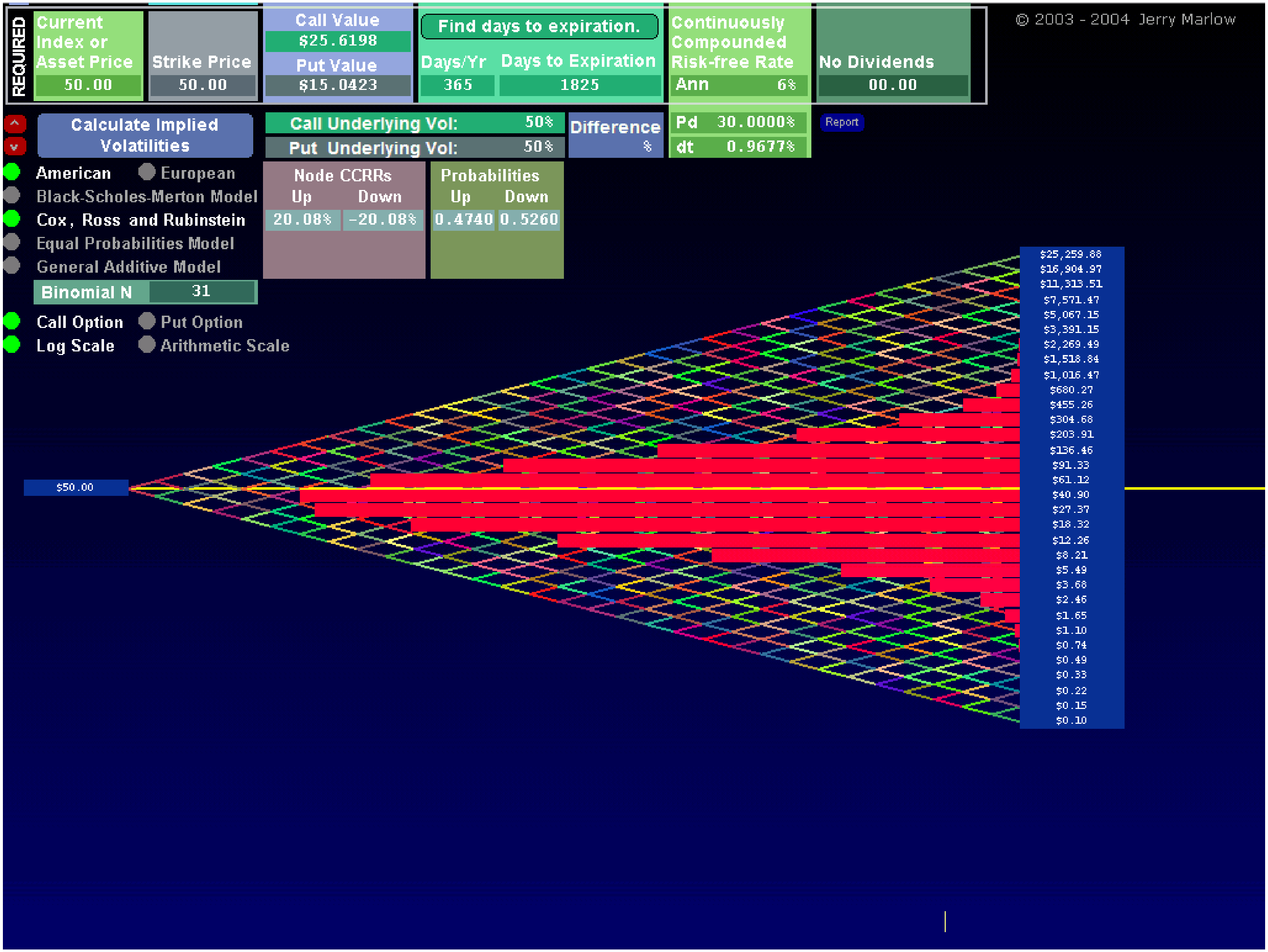


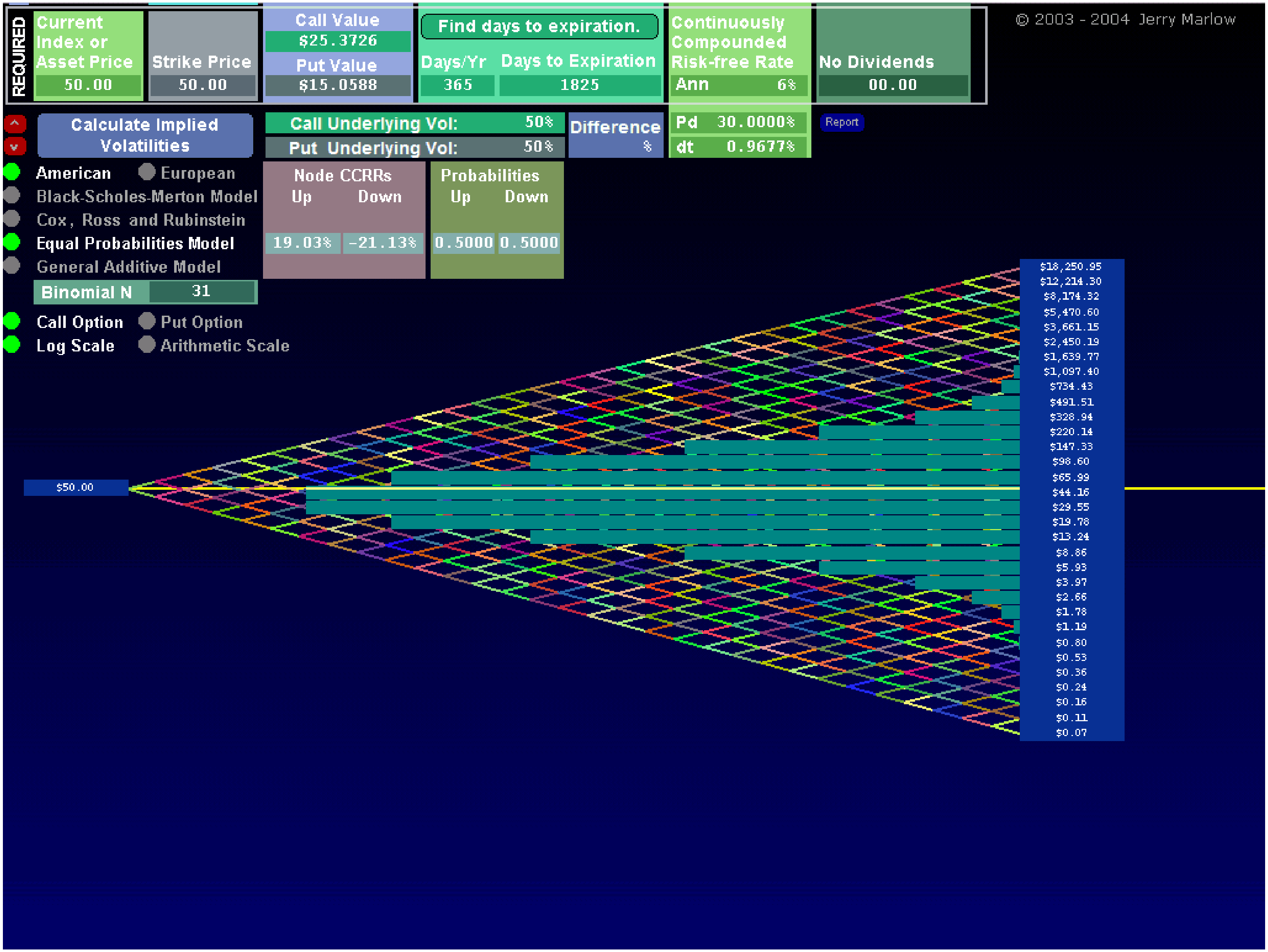
REQUIRED
Current Index or Asset Price
32.0000
Strike Price
40.0000
Call Value
$22.9865
Put Value
$12.1974
Find days to expiration.
Days/Yr
365
Days to Expiration
1825
Continuously Compounded Risk-free Rate
Ann 24.0227%
No Dividends
00.00
© 2003 - 2004 Jerry Marlow
Calculate Implied Volatilities
Call Underlying Vol: 69.3147%
Put Underlying Vol: 69.3147%
Difference
%
Pd 120.1135%
dt 10.0095%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 12
Node CCRRs
Up Down
44.74% -44.74%
Probabilities
Up Down
0.5000 0.5000
Call Option
Put Option
Log Scale
Arithmetic Scale
$6,869.40
$2,807.31
$1,147.26
$468.85
$191.61
$78.30
$32.00
$32.00
$13.08
$5.34
$2.18
$0.89
$0.36
$0.15

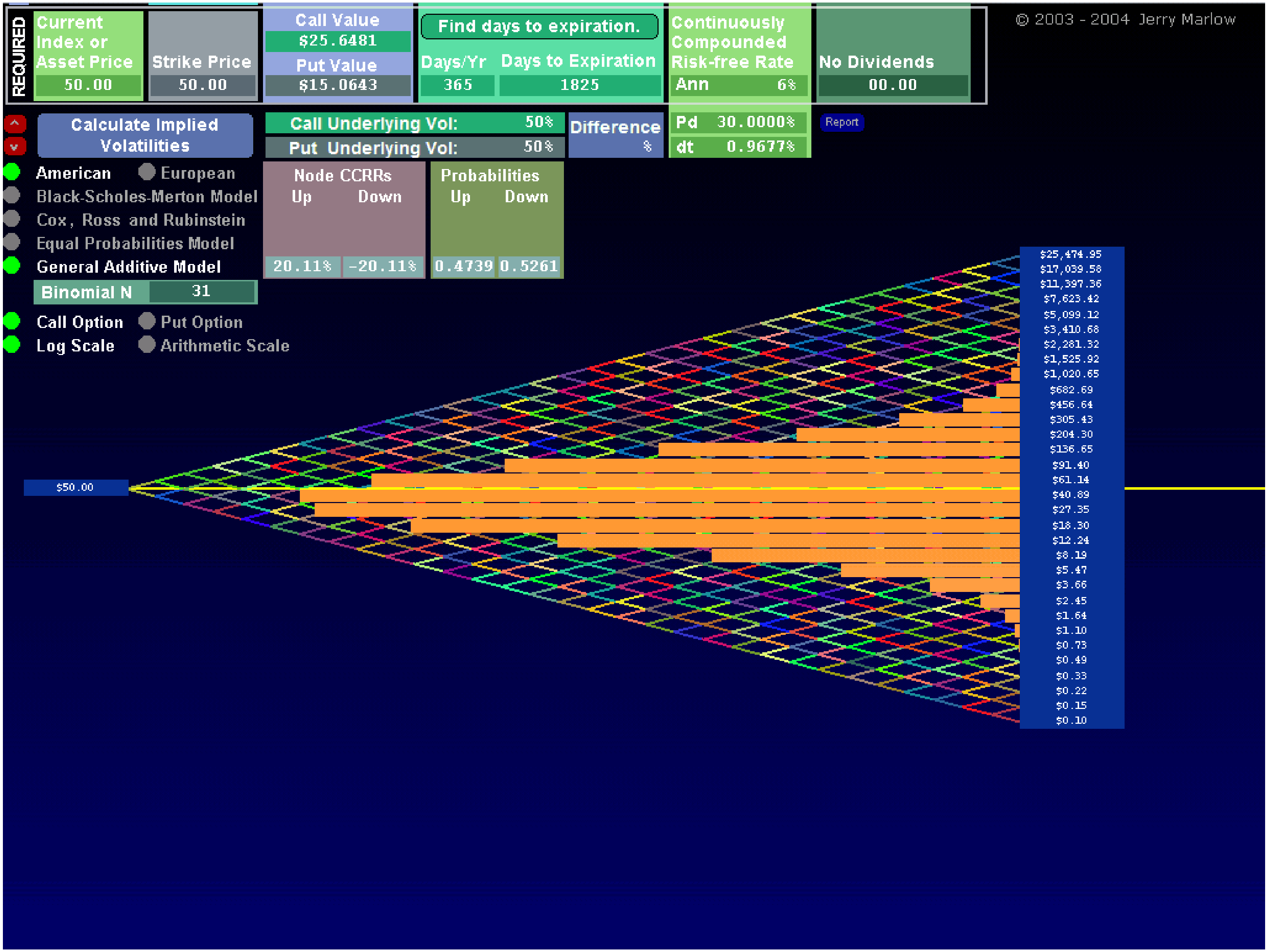
Subsequent illustrations give us a feel for why different models and different binomial Ns give different option values: they generate different probability distributions relative to the strike price.

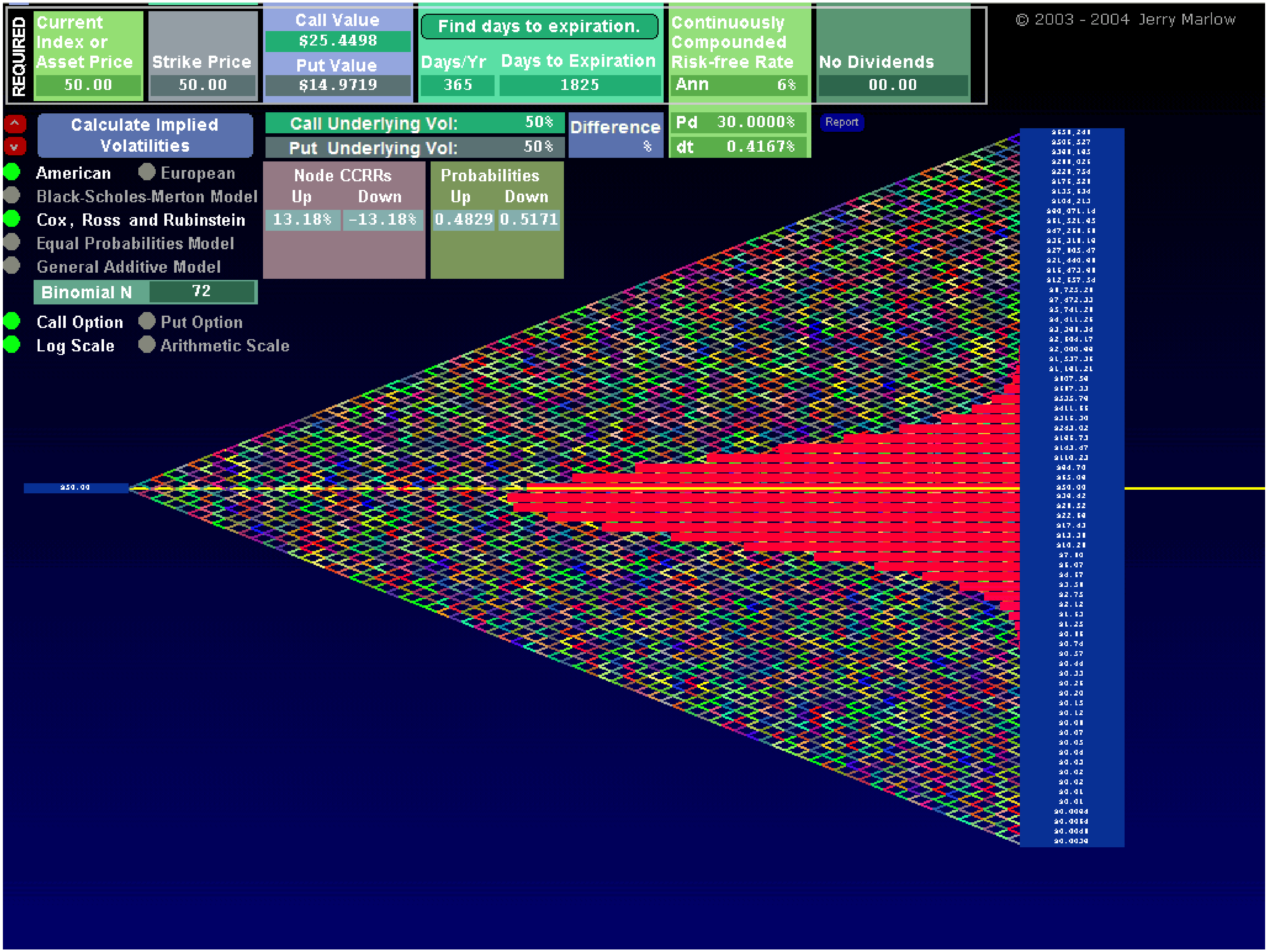
We look at the probability distributions of the different models for different binomial Ns for the second option whose values we graphed against N, the one that expires in 5 years, has a strike price of $50.00, whose underlying has a current stock price of $50.00 and an expected volatility of 50%. The risk-free rate is 6%. For the three models, we look at binomial Ns of 31 and 72.

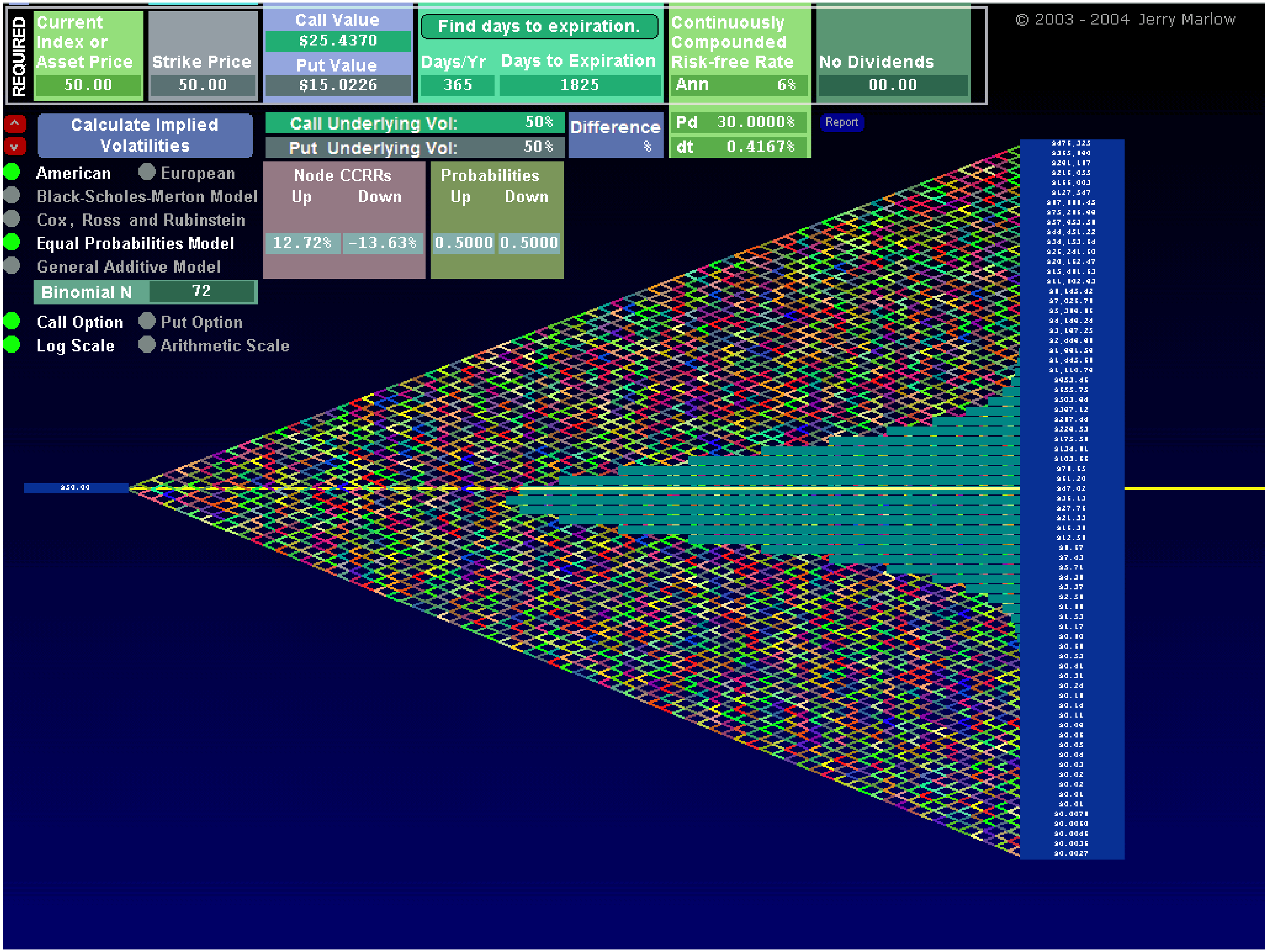
To facilitate visual comparison of these graphs, they are all drawn to the same scale. Where you see terminal stock prices without probability bars, there is some small probability of those outcomes occurring, but as drawn on a computer screen, the bars are less than one pixel wide. Hence they are not visible.

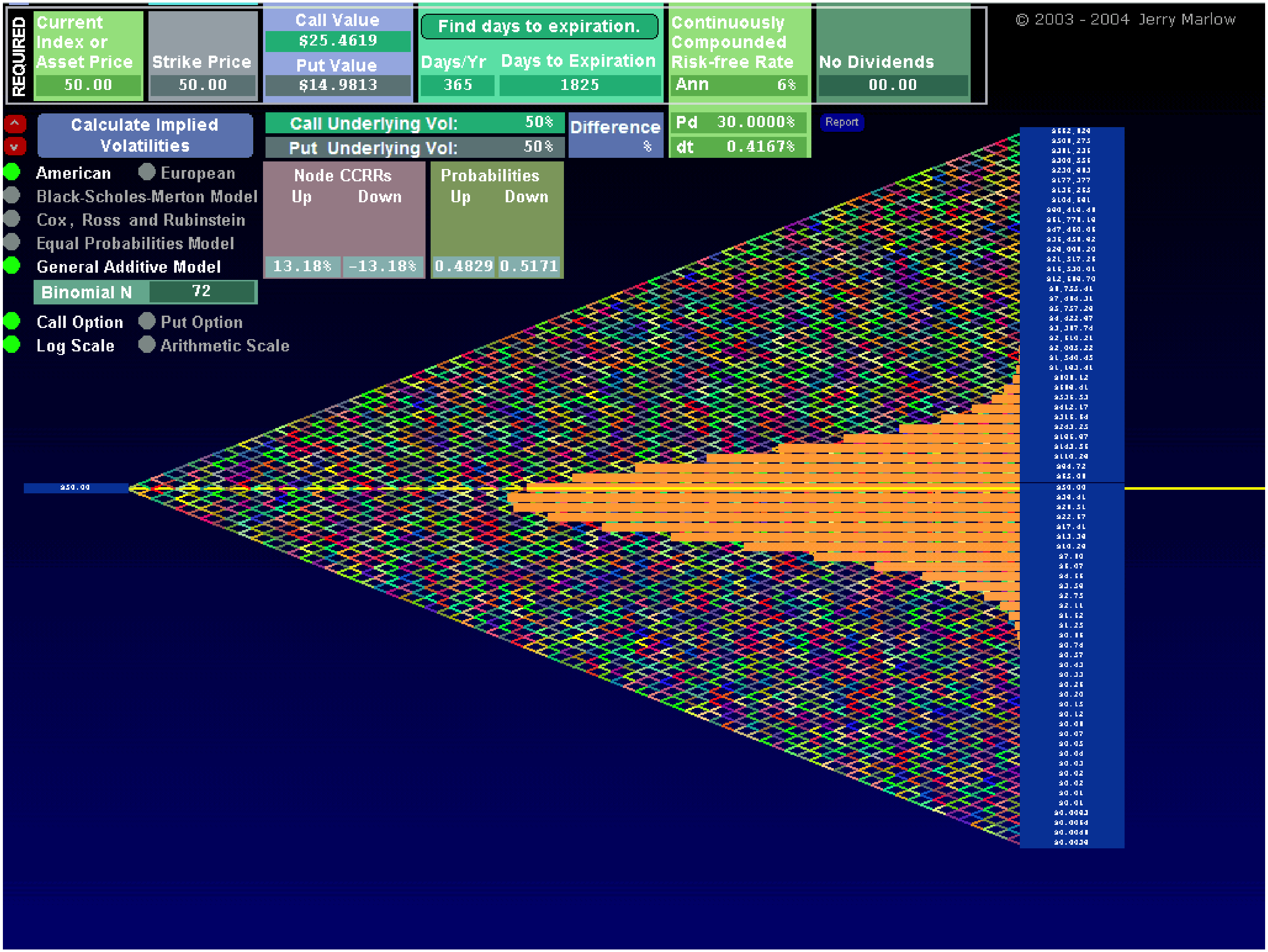

REQUIRED
Current Index or Asset Price
50.00
Strike Price
50.00
Call Value
$25.6198
Put Value
$15.0423
Find days to expiration.
Days/Yr
365
Days to Expiration
1825
Continuously Compounded Risk-free Rate
Ann 6%
No Dividends
00.00
© 2003 - 2004 Jerry Marlow
Calculate Implied Volatilities
Call Underlying Vol: 50%
Put Underlying Vol: 50%
Difference %
Pd 30.0000%
dt 0.9677%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 31
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
20.08% -20.08%
Probabilities
Up Down
0.4740 0.5260
$50.00
$25,259.88
$16,904.97
$11,313.51
$7,571.47
$5,067.15
$3,391.15
$2,269.49
$1,518.84
$1,016.47
$680.27
$455.26
$304.68
$203.91
$136.46
$91.33
$61.12
$40.90
$27.37
$18.32
$12.26
$8.21
$5.49
$3.68
$2.46
$1.65
$1.10
$0.74
$0.49
$0.33
$0.22
$0.15
$0.10


© 2003 - 2004 Jerry Marlow
REQUIRED
Current Index or Asset Price
50.00
Strike Price
50.00
Call Value
$25.3726
Put Value
$15.0588
Find days to expiration.
Days/Yr
365
Days to Expiration
1825
Continuously Compounded Risk-free Rate
Ann 6%
No Dividends
00.00
Calculate Implied Volatilities
Call Underlying Vol: 50%
Put Underlying Vol: 50%
Difference %
Pd 30.0000%
dt 0.9677%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 31
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
19.03% -21.13%
Probabilities
Up Down
0.5000 0.5000
$50.00
$18,250.95
$12,214.30
$8,174.32
$5,470.60
$3,661.15
$2,450.19
$1,639.77
$1,097.40
$734.43
$491.51
$328.94
$220.14
$147.33
$98.60
$65.99
$44.16
$29.55
$19.78
$13.24
$8.86
$5.93
$3.97
$2.66
$1.78
$1.19
$0.80
$0.53
$0.36
$0.24
$0.16
$0.11
$0.07


REQUIRED
Current Index or Asset Price
50.00
Strike Price
50.00
Call Value
$25.6481
Put Value
$15.0643
Find days to expiration.
Days/Yr
365
Days to Expiration
1825
Continuously Compounded Risk-free Rate
Ann 6%
No Dividends
00.00
© 2003 - 2004 Jerry Marlow
Calculate Implied Volatilities
Call Underlying Vol: 50%
Put Underlying Vol: 50%
Difference %
Pd 30.0000%
dt 0.9677%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 31
Node CCRRs Up Down
20.11% -20.11%
Probabilities Up Down
0.4739 0.5261
Call Option
Put Option
Log Scale
Arithmetic Scale
$50.00
$25,474.95
$17,039.58
$11,397.36
$7,623.42
$5,099.12
$3,410.68
$2,281.32
$1,525.92
$1,020.65
$682.69
$456.64
$305.43
$204.30
$136.65
$91.40
$61.14
$40.89
$27.35
$18.30
$12.24
$8.19
$5.47
$3.66
$2.45
$1.64
$1.10
$0.73
$0.49
$0.33
$0.22
$0.15
$0.10


REQUIRED
Current Index or Asset Price
50.00
Strike Price
50.00
Call Value
$25.4498
Put Value
$14.9719
Find days to expiration.
Days/Yr
365
Days to Expiration
1825
Continuously Compounded Risk-free Rate
Ann
6%
No Dividends
00.00
© 2003 - 2004 Jerry Marlow
Calculate Implied Volatilities
Call Underlying Vol: 50%
Put Underlying Vol: 50%
Difference
%
Pd 30.0000%
dt 0.4167%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Node CCRRs
Up Down
13.18% -13.18%
Probabilities
Up Down
0.4829 0.5171
Binomial N 72
Call Option
Put Option
Log Scale
Arithmetic Scale


REQUIRED
Current Index or Asset Price
50.00
Strike Price
50.00
Call Value
$25.4370
Put Value
$15.0226
Find days to expiration.
Days/Yr
365
Days to Expiration
1825
Continuously Compounded Risk-free Rate
Ann
6%
No Dividends
00.00
© 2003 - 2004 Jerry Marlow
Calculate Implied Volatilities
Call Underlying Vol:
50%
Put Underlying Vol:
50%
Difference
%
Pd
30.0000%
dt
0.4167%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Node CCRRs
Up
Down
12.72%
-13.63%
Probabilities
Up
Down
0.5000
0.5000
Binomial N
72
Call Option
Put Option
Log Scale
Arithmetic Scale


REQUIRED
Current Index or Asset Price
50.00
Strike Price
50.00
Call Value
$25.4619
Put Value
$14.9813
Find days to expiration.
Days/Yr
365
Days to Expiration
1825
Continuously Compounded Risk-free Rate
Ann 6%
No Dividends
00.00
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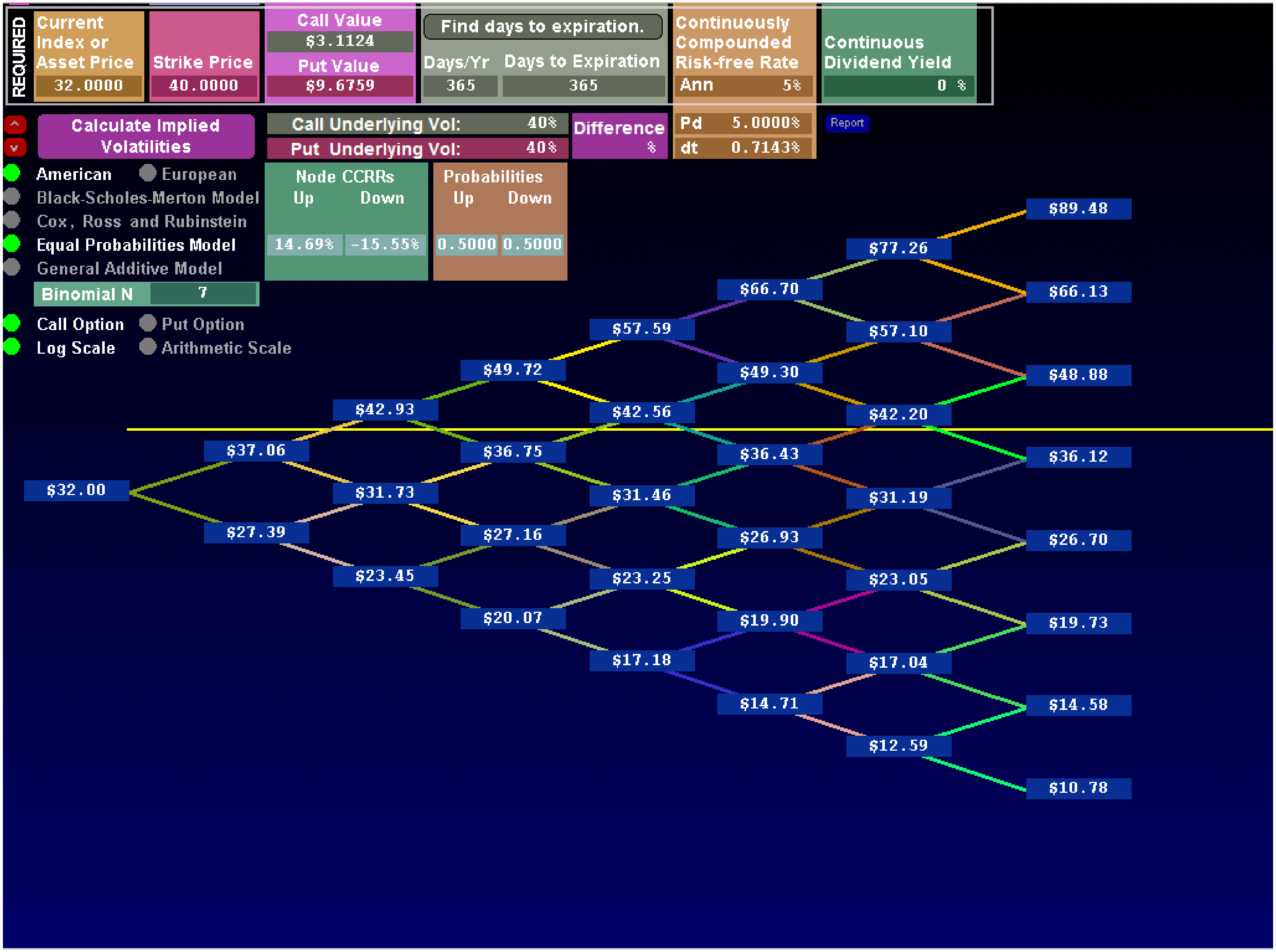
Calculate Implied Volatilities
Call Underlying Vol: 50%
Put Underlying Vol: 50%
Difference %
Pd 30.0000%
dt 0.4167%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Node CCRRs
Up Down
13.18% -13.18%
Probabilities
Up Down
0.4829 0.5171
Binomial N 72
Call Option
Put Option
Log Scale
Arithmetic Scale

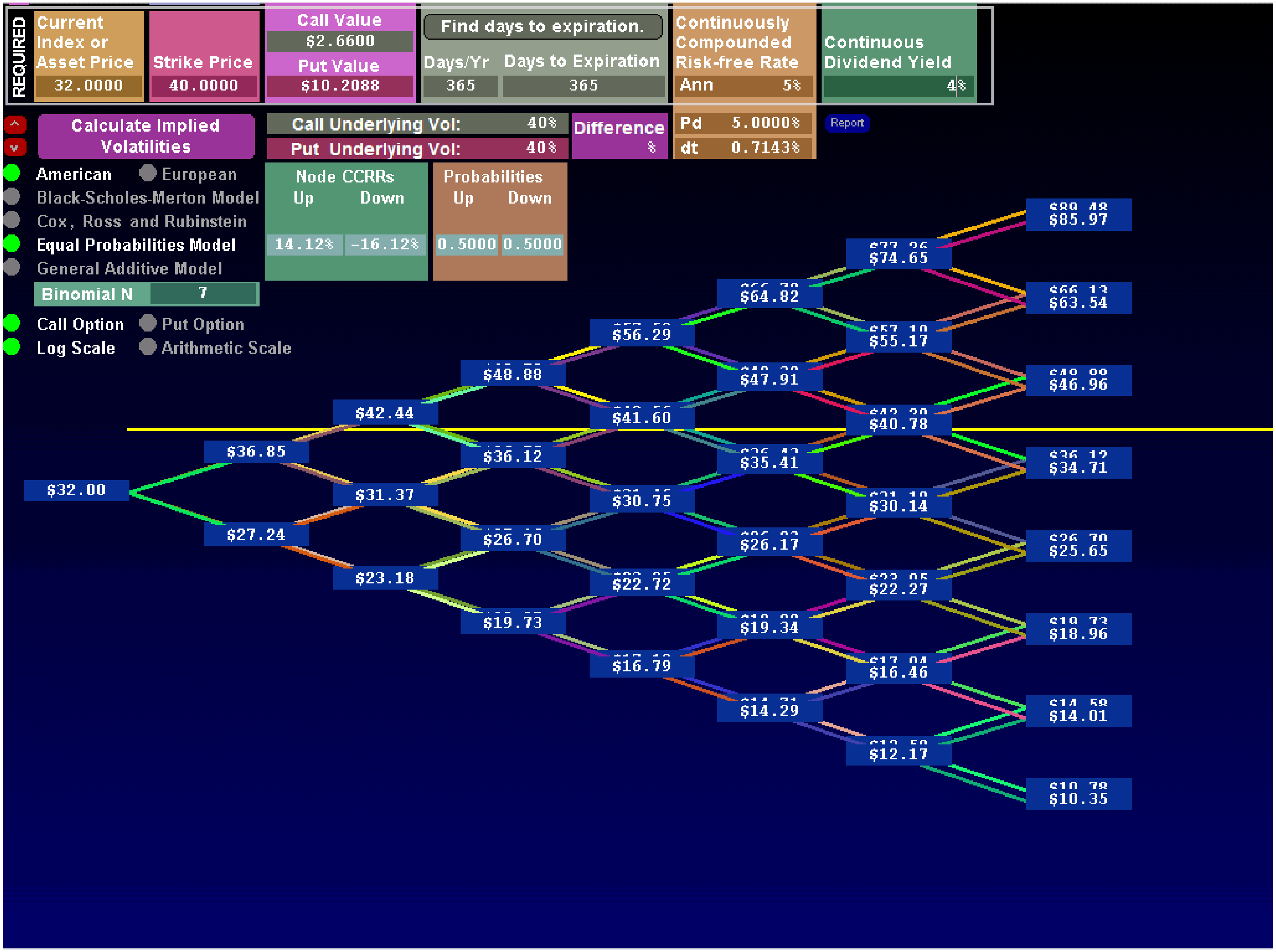
When an underlying pays a dividend yield, the price-path evolution is depressed below what it otherwise would be.

When we looked at the formulas that the different models use to calculate the interval up and down continuously compounded rates of return and probabilities, we saw that the formulas subtract any continuous dividend yield (q in the formulas) from the continuously compounded risk-free rate. In the Equal Probabilities and General Additive models, the effect of the subtraction is to depress the price-path evolution below what it would be if the underlying did not pay a dividend yield. In the Cox, Ross and Rubinstein model, the price-path evolution stays the same and the dividend yield changes the up and down probabilities.

If the underlying is expected to pay a dividend yield during a call option's time to expiration, the call has less value than it would if the underlying is not expected to pay a dividend. A put has a greater value if the underlying is expected to pay a dividend than it would if it is not.

The next illustration shows an Equal Probabilities model price-path evolution in which the underlying pays no dividend yield. The subsequent illustration overlays the price-path evolution when the underlying pays a 4% continuous dividend yield.


REQUIRED
Current Index or Asset Price
32.0000
Strike Price
40.0000
Call Value
$3.1124
Put Value
$9.6759
Find days to expiration.
Days/Yr
365
Days to Expiration
365
Continuously Compounded Risk-free Rate
Ann 5%
Continuous Dividend Yield
0 %
Calculate Implied Volatilities
Call Underlying Vol: 40%
Put Underlying Vol: 40%
Difference
%
Pd 5.0000%
dt 0.7143%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 7
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
14.69% -15.55%
Probabilities
Up Down
0.5000 0.5000
$32.00
$37.06
$27.39
$42.93
$31.73
$23.45
$49.72
$36.75
$27.16
$20.07
$57.59
$42.56
$31.46
$23.25
$17.18
$66.70
$49.30
$36.43
$26.93
$19.90
$14.71
$77.26
$57.10
$42.20
$31.19
$23.05
$17.04
$12.59
$89.48
$66.13
$48.88
$36.12
$26.70
$19.73
$14.58
$10.78


REQUIRED
Current Index or Asset Price
32.0000
Strike Price
40.0000
Call Value
$2.6600
Put Value
$10.2088
Find days to expiration.
Days/Yr
365
Days to Expiration
365
Continuously Compounded Risk-free Rate
Ann
5%
Continuous Dividend Yield
4%
Calculate Implied Volatilities
Call Underlying Vol: 40%
Put Underlying Vol: 40%
Difference
%
Pd 5.0000%
dt 0.7143%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 7
Call Option
Put Option
Log Scale
Arithmetic Scale
Node CCRRs
Up Down
14.12% -16.12%
Probabilities
Up Down
0.5000 0.5000
$32.00
$36.85
$27.24
$42.44
$31.37
$23.18
$48.88
$36.12
$26.70
$19.73
$56.29
$41.60
$30.75
$22.72
$16.79
$64.82
$47.91
$35.41
$26.17
$19.34
$14.29
$74.65
$55.17
$40.78
$30.14
$22.27
$16.46
$12.17
$85.97
$63.54
$46.96
$34.71
$25.65
$18.96
$14.01
$10.35

Payment of quarterly or other lumpy dividends depresses price paths at the time of the payments.

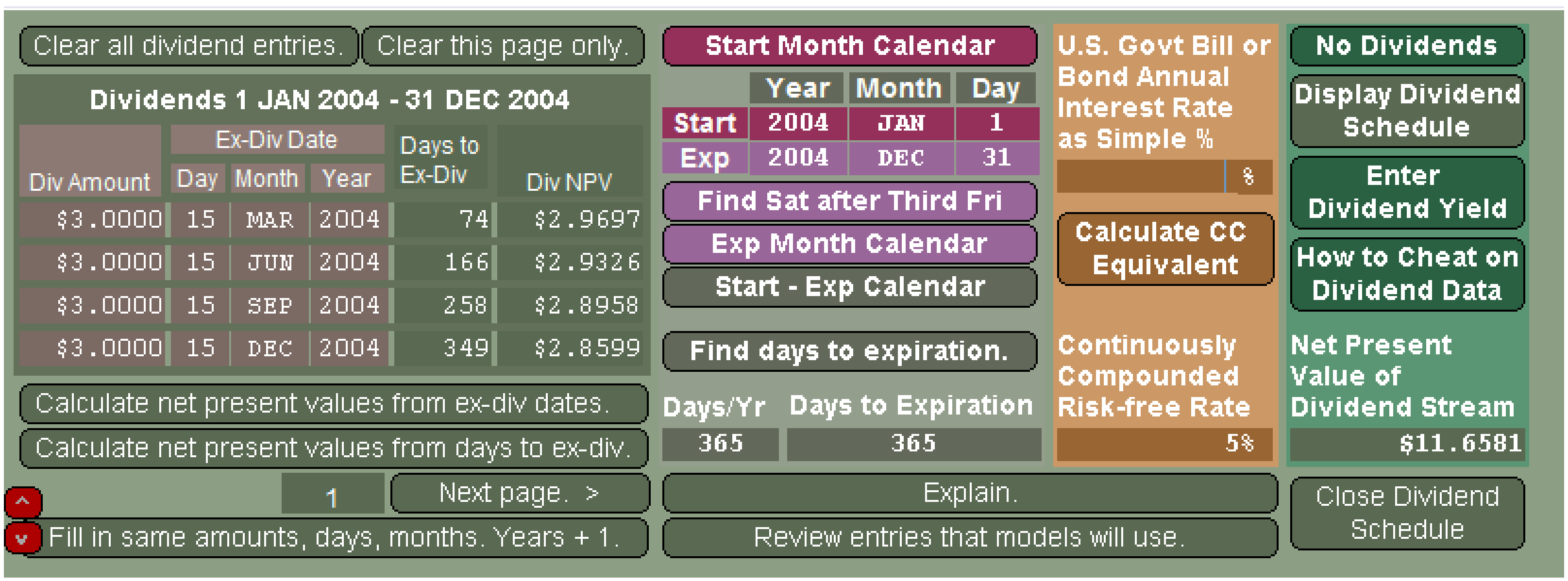
When a stock or other underlying makes a dividend payment, the value of the underlying is reduced by the amount of the dividend. Consequently, when the underlying goes ex-dividend, the value of the underlying drops by the amount of the dividend.

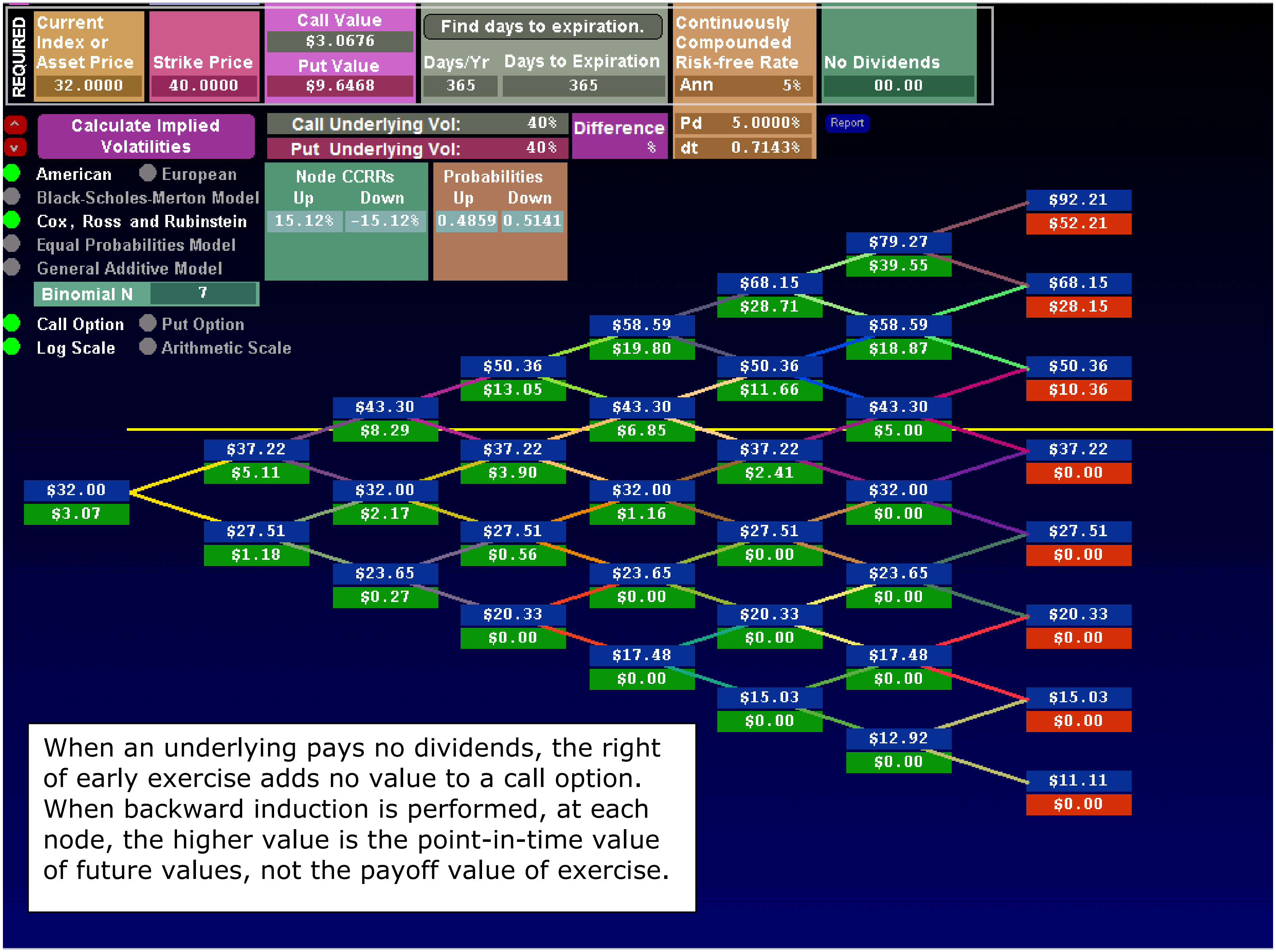
To model the effect of the payment of lumpy dividends on stock price paths, binomial models usually treat the value of the underlying as consisting of two components: a non-volatile amount that will grow at the risk-free rate and be used to fund dividend payments and a volatile amount that reflects the uncertain value of the underlying. To implant this strategy, models follow this procedure:

1. Calculate the present value of each dividend that the underlying is expected to pay during the option's time to expiration.
2. Sum the dividend present values.
3. Subtract the sum from the current market price of the underlying. Use this amount to fund the dividend payments.
4. At each node, add the value of unpaid dividends plus interest earned to the node value of the underlying.

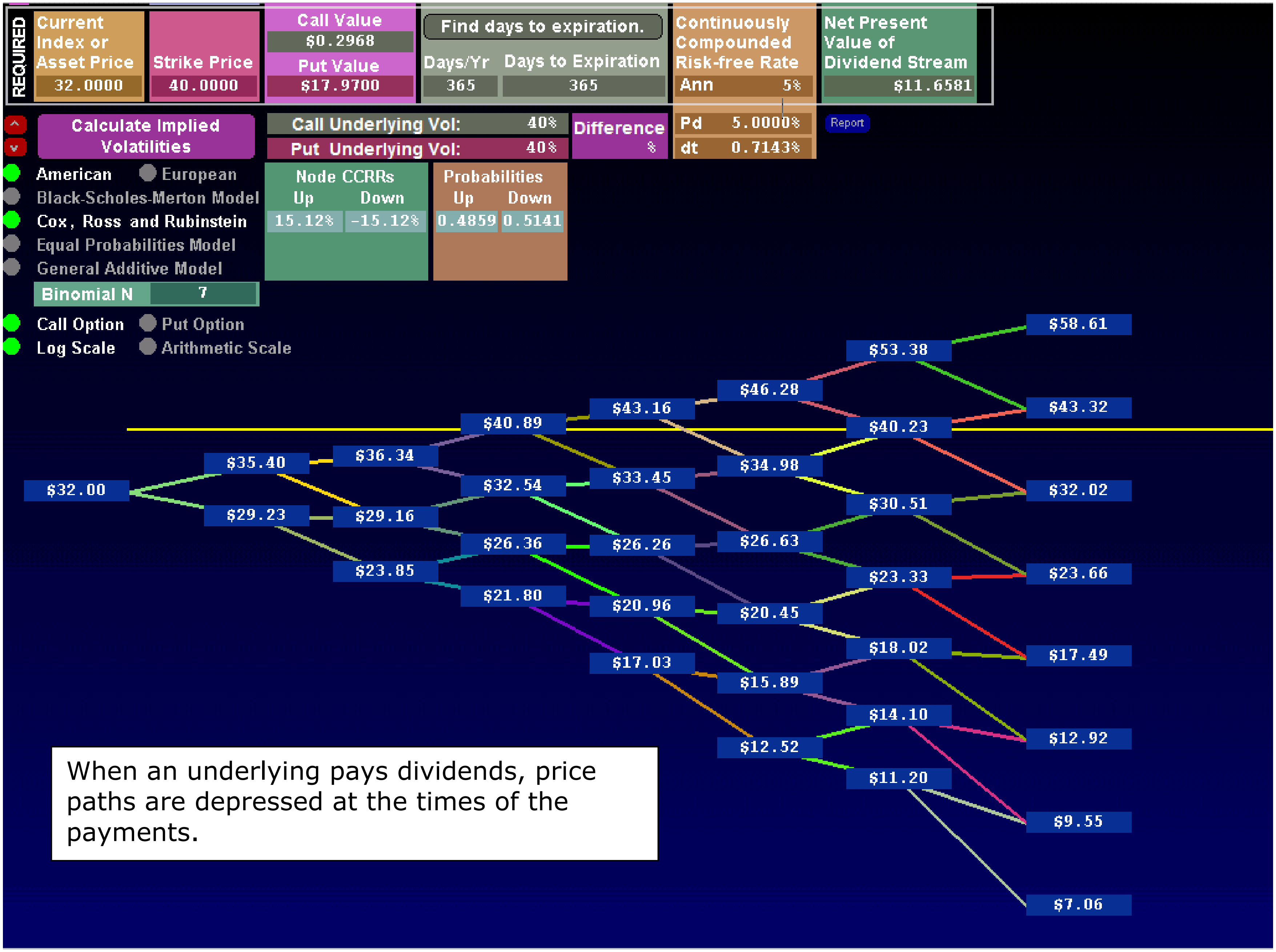
When models use this strategy, dividend payments depress price paths at the time the dividends are paid.

To illustrate, we compare two price-path evolutions: one with and one without four overly large dividend payments.

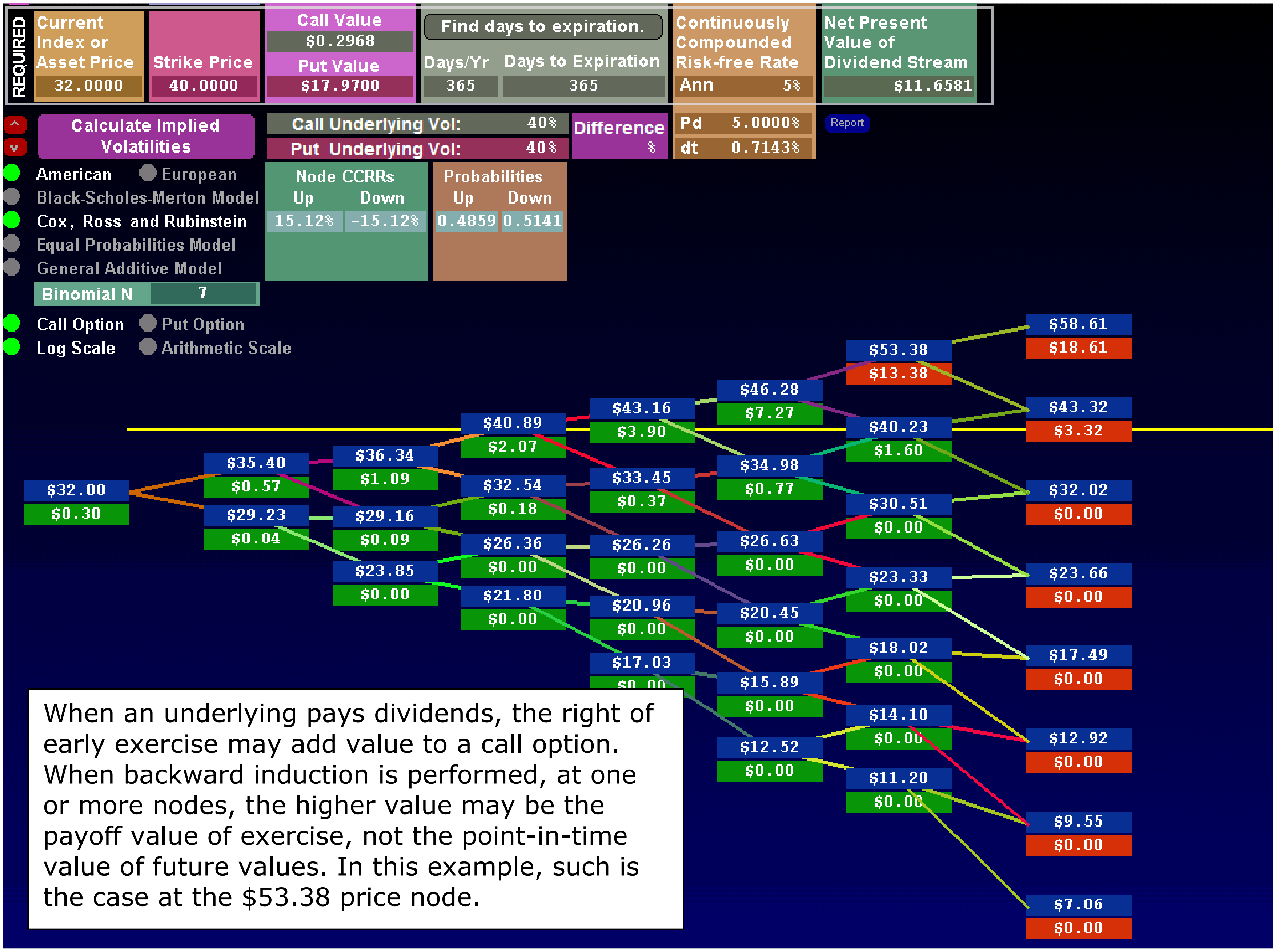




When an underlying pays no dividends, the right of early exercise adds no value to a call option. When backward induction is performed, at each node, the higher value is the point-in-time value of future values, not the payoff value of exercise.



When an underlying pays dividends, price paths are depressed at the times of the payments.



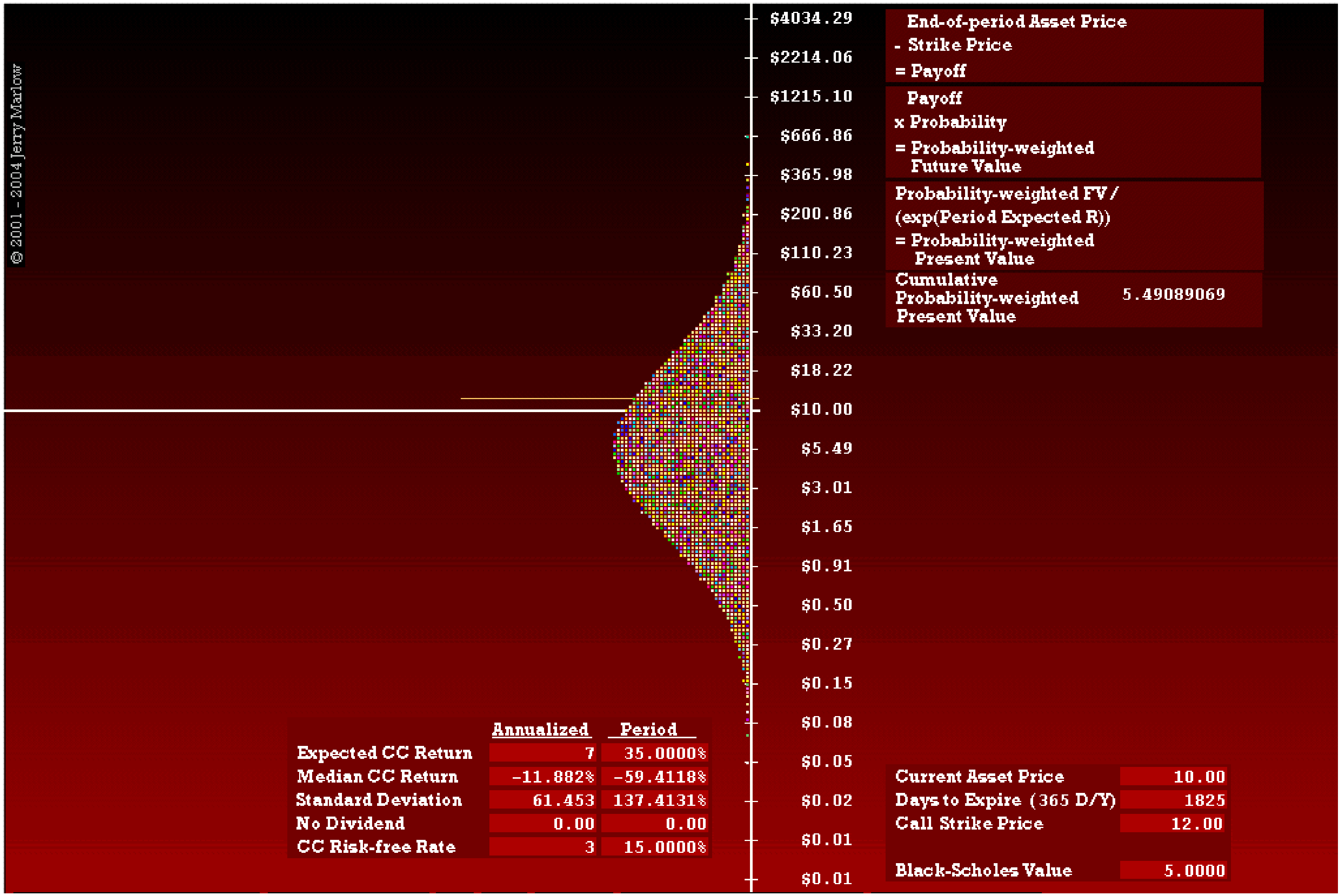
When an underlying pays dividends, the right of early exercise may add value to a call option. When backward induction is performed, at one or more nodes, the higher value may be the payoff value of exercise, not the point-in-time value of future values. In this example, such is the case at the $53.38 price node.

If investors in a stock are risk averse, then the stock offers a risk premium

In the preceding discussions, to make an option's probability-weighted present value equal its Black-Scholes or arbitrage-free value, we set the expected return of the stock forecast equal to the risk-free rate. But is a stock's expected return equal to the risk-free rate?

Probably not. Intuitively, most people probably, usually expect a stock on average to pay a higher return than the risk-free rate that they could earn on a U.S. government bond. Otherwise only gamblers would buy stocks instead of risk-free investments.

Economists generally assume that investors are risk averse. Since the financial markets are auctions, to the degree that investors are risk averse, they bid less for the uncertain future cash flows of risky investments than they bid for the certain future cash flows of risk-free investments. The lower prices that investors bid for risky investments mean that risky investments offer an expected return higher than the risk-free rate. Risky investments offer an expected return of the risk-free rate *plus* a risk premium. (Empirically we cannot figure out what the risk premium is because we do not know what the *true* forecast is and we do not know what forecasts investors have in mind.)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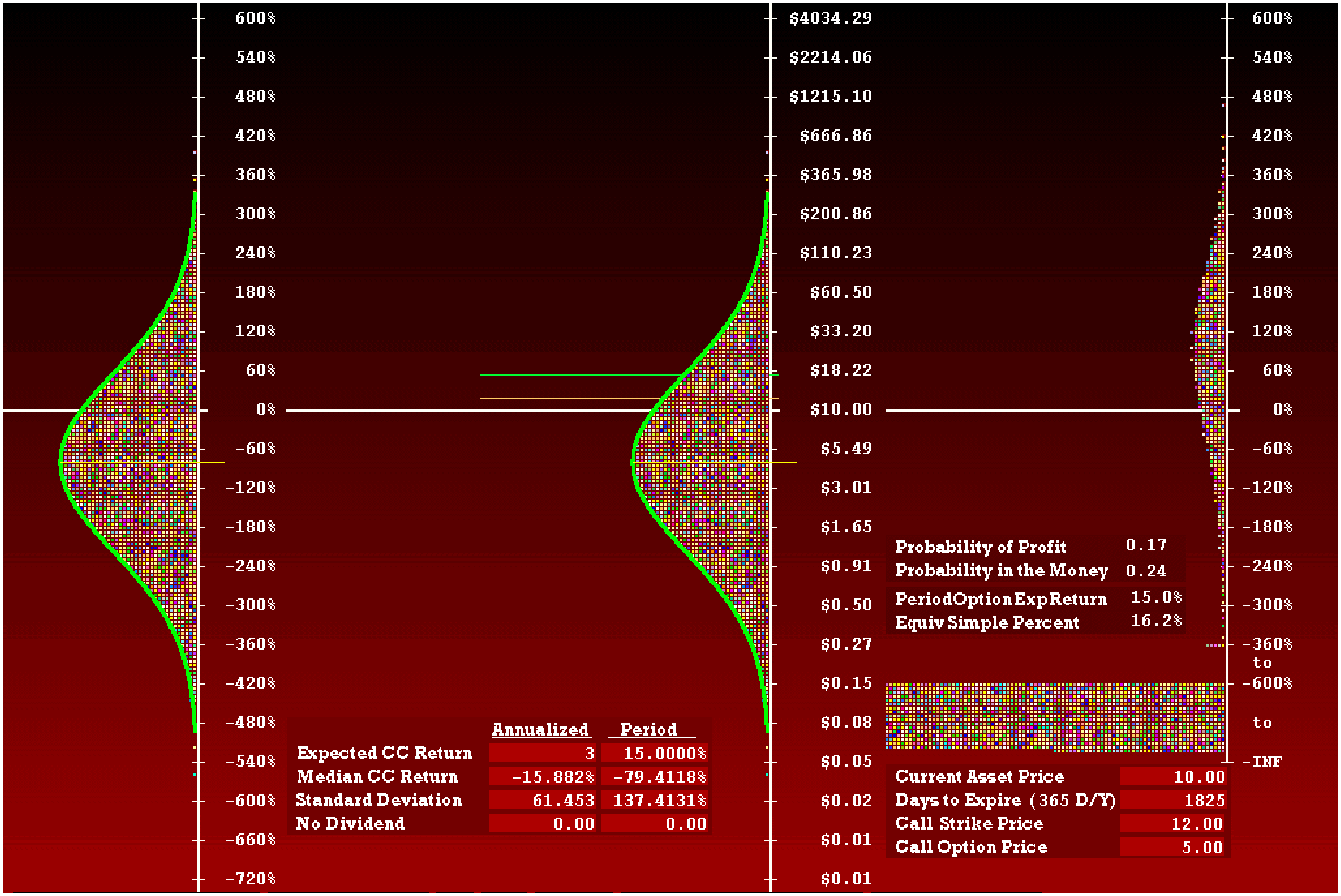
$4034.29
$2214.06
$1215.10
$666.86
$365.98
$200.86
$110.23
$60.50
$33.20
$18.22
$10.00
$5.49
$3.01
$1.65
$0.91
$0.50
$0.27
$0.15
$0.08
$0.05
$0.02
$0.01
$0.01
End-of-period Asset Price
- Strike Price
= Payoff
Payoff
x Probability
= Probability-weighted Future Value
Probability-weighted FV / (exp(Period Expected R))
= Probability-weighted Present Value
Cumulative Probability-weighted Present Value 5.49089069
Annualized Period
Expected CC Return 7 35.0000%
Median CC Return -11.882% -59.4118%
Standard Deviation 61.453 137.4131%
No Dividend 0.00 0.00
CC Risk-free Rate 3 15.0000%
Current Asset Price 10.00
Days to Expire (365 D/Y) 1825
Call Strike Price 12.00
Black-Scholes Value 5.0000

If a stock offers a risk premium, then the probability-weighted present value of a call option written on that stock is greater than its Black-Scholes or risk-neutral value

To get an option's probability-weighted present value to equal its Black-Scholes value, we set the expected return of the stock forecast equal to the risk-free rate. If, as in the calculations shown here, we include a risk premium in the stock forecast, then we get a higher probability-weighted present value for the option. If, in the stock forecast, we add a risk premium of 4% to the risk-free rate of 3%, then we go from a Black-Scholes or risk-neutral valuation of $5.00 for the option to a probability-weighted present value of $5.49.


600%
540%
480%
420%
360%
300%
240%
180%
120%
60%
0%
-60%
-120%
-180%
-240%
-300%
-360%
-420%
-480%
-540%
-600%
-660%
-720%
$4034.29
$2214.06
$1215.10
$666.86
$365.98
$200.86
$110.23
$60.50
$33.20
$18.22
$10.00
$5.49
$3.01
$1.65
$0.91
$0.50
$0.27
$0.15
$0.08
$0.05
$0.02
$0.01
$0.01
Annualized Period
Expected CC Return 3 15.0000%
Median CC Return -15.882% -79.4118%
Standard Deviation 61.453 137.4131%
No Dividend 0.00 0.00
Probability of Profit 0.17
Probability in the Money 0.24
PeriodOptionExpReturn 15.0%
Equiv Simple Percent 16.2%
-360% to -600% to -INF
Current Asset Price 10.00
Days to Expire (365 D/Y) 1825
Call Strike Price 12.00
Call Option Price 5.00

Under risk-neutral valuation, a stock and an option on that stock have the same expected return.

In the beginning, when we simulated option payoffs for different price paths, the software also calculated the continuously compounded rate of return an investor would have earned had he or she invested in the option. For example, if the option cost $5.00 and the payoff was $5.81, then the continuously compounded rate of return of investing in the option would be:

CCRR = log(Payoff / Option Price)

= log($5.81/$5.00)

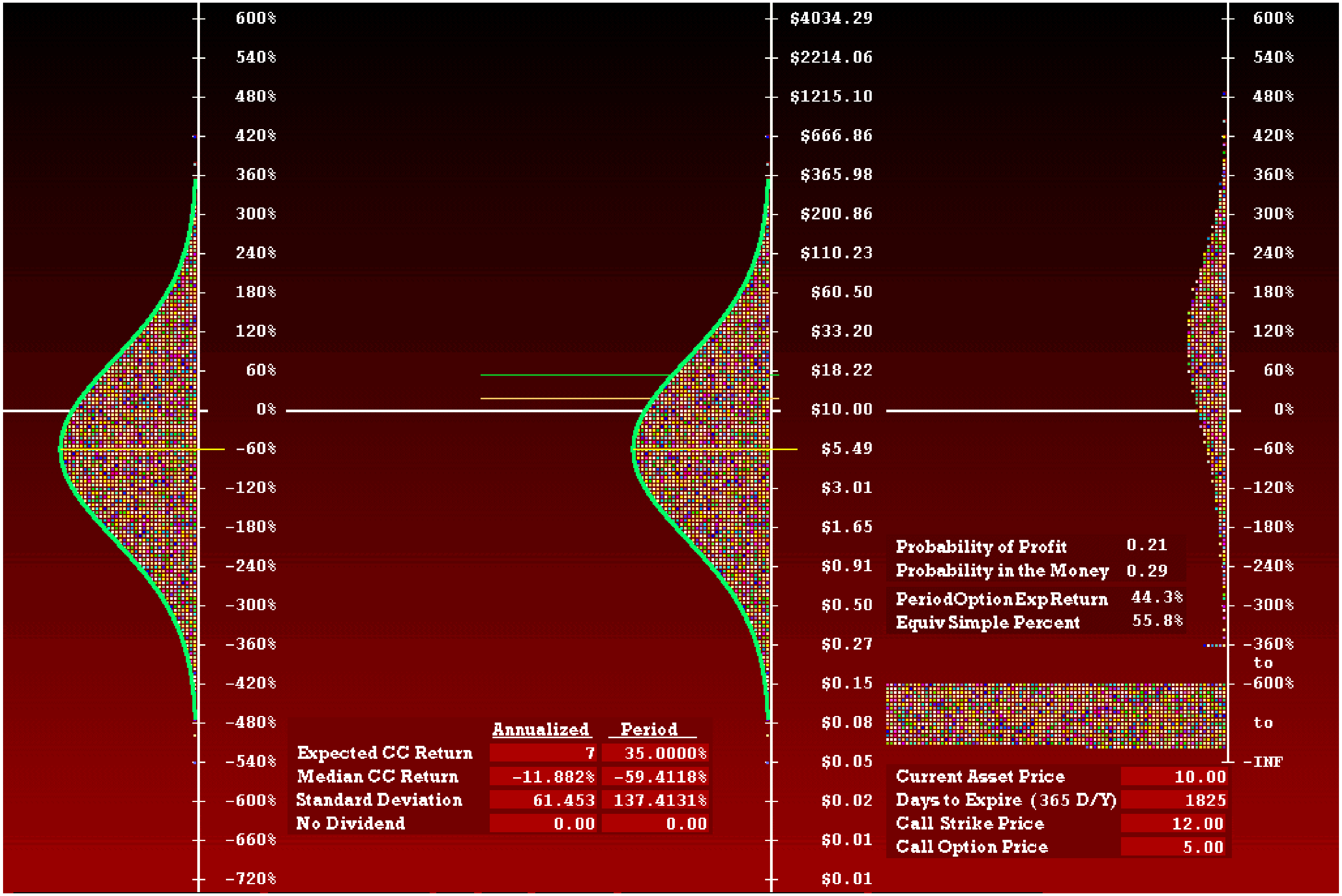
= 15%.

How might we calculate an option's forecast average return or expected return?

To calculate the option's risk-neutral value, we drew the stock forecast as a bell-shaped curve, stepped through it, calculated the probability-weighted present value of each payoff, and summed the values. Using a similar process, we can calculate the average or expected return of investing in the option.

Here we step through the stock's bell-shaped curve and, with a little square, plot the continuously compounded rate of return of each option payoff. We keep a running average of the option returns.

When we set the stock forecast's expected return equal to the risk-free rate, we find that the option forecast's expected return also equals the risk-free rate. In the illustration, for the five-year investment period, both the stock and the option have an average forecast return or expected return of 15%.


600%
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420%
360%
300%
240%
180%
120%
60%
0%
-60%
-120%
-180%
-240%
-300%
-360%
-420%
-480%
-540%
-600%
-660%
-720%
$4034.29
$2214.06
$1215.10
$666.86
$365.98
$200.86
$110.23
$60.50
$33.20
$18.22
$10.00
$5.49
$3.01
$1.65
$0.91
$0.50
$0.27
$0.15
$0.08
$0.05
$0.02
$0.01
$0.01
-360% to -600% to -INF
Annualized Period
Expected CC Return 7 35.0000%
Median CC Return -11.882% -59.4118%
Standard Deviation 61.453 137.4131%
No Dividend 0.00 0.00
Probability of Profit 0.21
Probability in the Money 0.29
PeriodOptionExpReturn 44.3%
Equiv Simple Percent 55.8%
Current Asset Price 10.00
Days to Expire (365 D/Y) 1825
Call Strike Price 12.00
Call Option Price 5.00

If a stock offers a risk premium, then a call option written on that stock leverages the risk premium.

If we include a risk premium in the stock's expected return, then a call option's expected return is not only higher than the risk-free rate, it is also higher than the expected return of the stock. In other words, if a stock offers a risk premium, a call option written on that stock leverages the risk premium. Here, with the risk-free rate equal to 15% for the five-year investment period and the stock's expected return equal to 35%, the option's expected return is equal to 44.3%.

"So what?," you might ask.

When a court values options, most likely some of the back and forth discussion can best be viewed as arguments of whether to use a no-arbitrage approach to valuation or an expectations approach. If someone argues that the Black-Scholes or risk-neutral value is too high because he or she does not expect the stock price to go very high, a counter to that argument is that the Black-Scholes or risk-neutral valuation is actually quite conservative: the Black-Scholes or risk-neutral value leaves the risk premium out of the stock forecast and the leveraging of the risk premium out of the option forecast.

Risk-neutral valuation is a way to come up with arbitrage-free values for options. It does not mean investors are risk neutral.

Risk-neutral valuation doesn't say that investors are risk neutral. It is a way of coming up with arbitrage-free, market-equilibrium values for options. Risk-neutral valuation leaves investors in stocks with their risk premiums. It leaves the owners of call options with their leveraged risk premiums— whatever they may be.

The financial world relies upon Black-Scholes and risk-neutral-valuation models

Financial institutions around the world use Black-Scholes options pricing theory and risk-neutral-valuation methodologies to value and price not only stock options but many different types of options. These methodologies give values for options that make the equilibrium prices of all options consistent with one another.

The marketplace enforces risk-neutral pricing of options. Would-be arbitragers lurk in the financial markets. They are on the lookout for options that have prices that are inconsistent with the price volatilities of underlying securities. If they can buy under-priced options or sell over-priced options and dynamically hedge their transactions, then they can earn risk-free profits. Risk-neutral valuation gives option prices that are not susceptible to arbitrage. Arbitrage drives the prices of all options toward their risk-neutral-values.

In general, to the degree that the financial markets are efficient and complete and trading is continuous, traders, speculators, and would-be arbitragers can hedge the sale of just about any security. Hence, in risk-neutral valuation, the arbitrage-free expected return of any security is the risk-free rate.

Reality does not conform exactly to the Black-Scholes or binomial-modeling theories

Market prices of options do not conform exactly to Black-Scholes options pricing theory or to binomial pricing models. The underlying theories make a number of assumptions about marketplace dynamics but the assumptions do not capture the dynamics exactly.

In particular, theories based on geometric Brownian motion assume that the probability distribution of continuously compounded rates of return at the end of any investment horizon is normally or binomially distributed. The relationships among option prices in the marketplace suggest that market participants believe that the actual distribution has thicker tails than does a normal or binomial distribution.

Also, Black-Scholes option pricing theory and binomial valuation methods assume that a stock's price volatility will remain the same over a given investment horizon. Some market participants argue that as stock prices change, their volatility changes.

Sophisticated financial firms modify Black-Scholes and the binomial models and design and build their own risk-neutral-valuation models to take into account the ways in which they believe marketplace dynamics diverge from the assumptions that underlie Black-Scholes and standard binomial models.

Even so, Black-Scholes options pricing theory and binomial models provide an effective way to tie together the effect on an option's value of the stock price, expected dividends from the underlying, the risk-free rate, the option's strike price, the stock's expected future volatility, and the option's time to expiration. In valuing employee stock options, we are looking for ways to come up with values that are consistent with the values of market-traded options. Black-Scholes and binomial models serve this purpose well. We can modify our application of these models to take into account how market prices diverge from strict adherence to the models.

Black-Scholes and risk-neutral valuation offer courts the fairest methodology with which to value employee stock options in divorce proceedings

The overwhelming reliance by the financial industry on risk-neutral and Black-Scholes valuation methodologies and the enforcement of risk-neutral-values in the marketplace argue strongly for courts to base their valuation of employee stock options on these methodologies.

Risk-neutral valuation brings the value of any option into line with the market price of the underlying stock, with the uncertainty associated with the stock's future price, and with the values of all other options. It provides courts with a way to assign values to employee stock options that are consistent with the values of market-traded options. By not factoring the risk premium into the option's probability-weighted present value, risk-neutral valuation compensates the spouse who takes possession of the option for his or her continuing exposure to risk.

Valuing employee stock options requires complete information

To apply Black-Scholes or risk-neutral valuation to employee stock options, for each option grant, a valuation expert needs a complete set of information. The information that must come from the requesting attorney includes the following:

- Grant date
- Vesting date or dates
- Valuation as-of date or dates
- Expiration date or dates of options in grant
- History of any stock splits since the grant date
- Number of options in grant and how number has been adjusted for any stock splits
- Option exercise price in grant and how exercise price has been adjusted for any stock splits
- If the options are not yet vested, the valuation expert will need from the requesting attorney a statement of the geometric average rate of the company's employee turnover and a statement of the frequency and extent to which the company waives the vesting requirements stated in the grant document and confers options upon departing employees.
- If the grant stipulates any conditions and/or requirements that may affect the employee's right to exercise the options after they are vested, then the valuation expert will need a statement of those conditions and/or requirements. If the conditions or requirements pertain to the employee's continued employment, then the valuation expert will need a statement of the geometric average rate of the company's employee turnover.

From public sources, the valuation expert should be able to obtain the following information and estimates necessary to perform the valuation:

- Market price of the stock or split-adjusted market price as of the valuation date
- Estimate of expected future stock dividends
- Risk-free rate that corresponds to options' time to expiration
- Prices of market-traded options on the company's stock from which to derive an estimate of the stock's future volatility

An example: Value 5,000 XYZ options

In the following discussions, we will use the same example: Value 5,000 XYZ call options that have an exercise or strike price of $14.00 and expire on December 7, 2010. For the purpose of illustration, we will vary other characteristics of the options.

Selection of a valuation as-of date is a question of law

The market price of an option's underlying stock may change significantly from day to day. As the price of the stock changes, the value of the option changes. Hence, selection of a valuation as-of date may affect significantly the calculated value of the options.

Selection of the valuation date also determines the time to expiration that goes into the value calculation. The longer an option's time to expiration, the greater its value.

An option-valuation expert can value an option from any as-of date up until and including the present day, but the selection of a valuation date is a question of law. Candidates for valuation date include: filing date, separation date, dissolution date, settlement date, and current date. The selection of and rationale for the selection of the valuation date may be different in community-property and non-community-property states.

A possible argument for choosing the latest feasible valuation date is that the purpose of risk-neutral valuation is to value an option in the face of uncertain future events. To the extent that the passage of time has resolved some of that uncertainty, it has rendered probabilistic methods unnecessary. Using a valuation date in the past forces parties to pretend that uncertainty that no longer exists still exists. Hence, selection of a valuation as-of date in the past— depending on how the stock price has changed since the valuation date— may seem to give advantage unfairly to one spouse or the other. Choosing the present day or the latest feasible valuation as-of date may be seen as being more in the spirit of keeping the options in a constructive trust up until the valuation as-of date.

If the attorney who requests the valuation is not certain which date the court will choose as a valuation as-of date, he or she may wish to have the expert value the options as-of several different dates.

For our XYZ options, we will use a valuation date of October 29, 2003.

Employee stock options may be identical to market-traded options or may differ from them in several respects: strike price, time to expiration, in not being vested, and in having conditions attached to their exercise.

A market-traded call option is a contract. Up until its expiration, the contract gives its owner the right to buy the underlying stock at a stated strike price. The contract has no conditions attached to it. Ordinarily the contractual rights are irrevocable.

For an employee stock option to be identical to a market-traded call option, it has to have the same strike price and same time to expiration as the market-traded option. The employee's right to exercise the option up until its expiration has to be unconditional and irrevocable. In practical terms, this requirement means that the option has to be vested and the time period during which the employee has the right to exercise the option cannot depend upon the employee's continued employment by the granting company.

In turn, we look at ways to value employee stock options that are identical to market-traded options, that differ in strike price, that differ in time to expiration, that are not yet vested, and that have conditions attached to their exercise.

If options identical to an employee stock option are available in the marketplace, then the market price is the fair value .

If an employee stock option is identical to a market-traded option in strike price, time to expiration and contractual rights then one simply can take the market price of the market-traded option to be the fair value of the employee stock options.

If employee stock options are otherwise identical to market-traded options but have a different strike price or time to expiration, then either Black-Scholes or risk-neutral models can be used to value the options.

To value an option using Black-Scholes or risk-neutral valuation, one needs to know:

- Market price of stock as of the valuation date
- Expected future stock dividends
- Strike price of option
- Expiration date of option
- Applicable continuously compounded risk-free rate
- Expected future volatility of stock

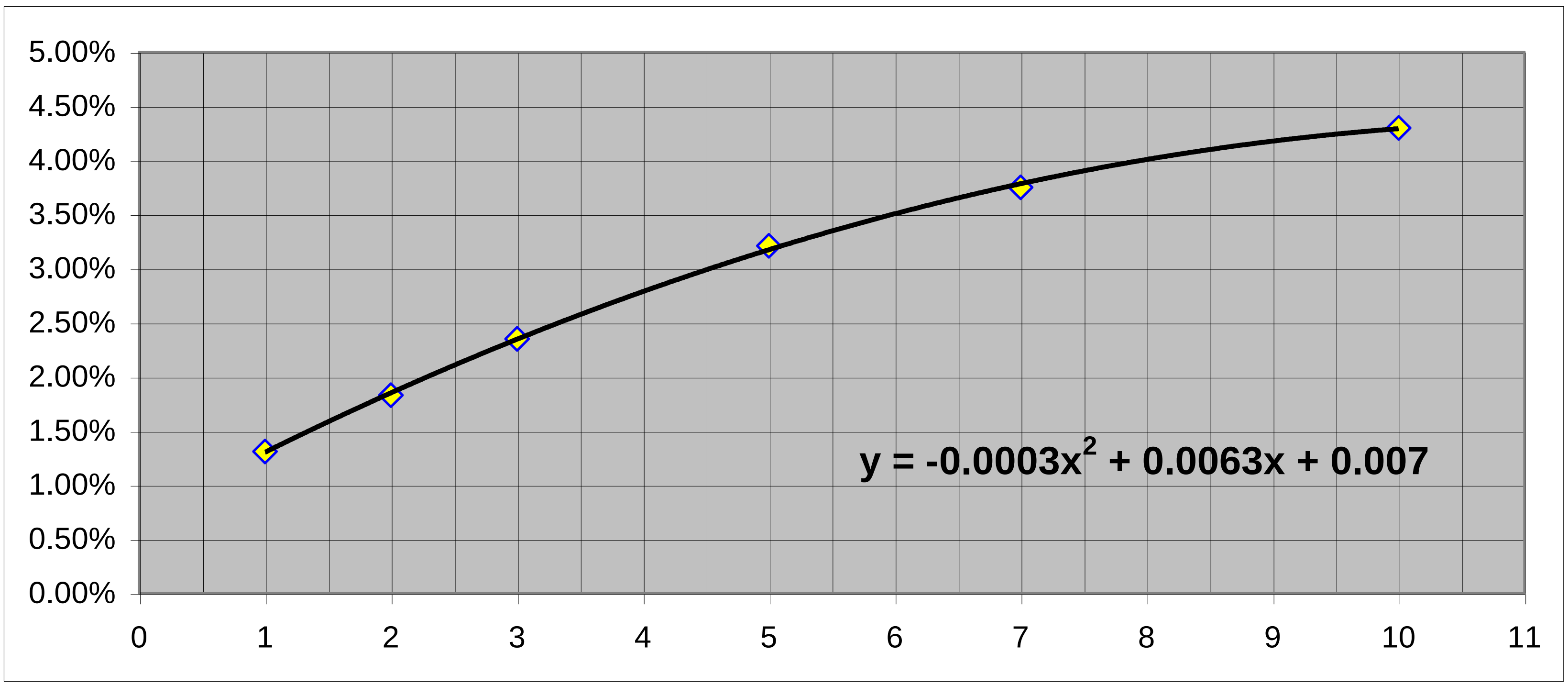
All of these values are straightforward except the applicable continuously compounded risk-free rate and the expected future volatility of the stock.

To value options, use the continuously compounded risk-free rate for a term equal to the option's time to expiration.

Usually short-term interest rates differ from long-term rates. To calculate option values, use the rate paid by risk-free bills or bonds that have a time to maturity equal to or approximately equal to the option's time to expiration. One way to arrive at such a rate is to follow these steps:

1. Obtain current or as-of-valuation-date U.S. government bond rates.
2. Use spreadsheet software to fit a line to the data.
3. Use the equation for the line to calculate a simple interest rate for the option's time to expiration.
4. Convert the simple rate to a continuously compounded rate of return.

The chart below is a plot of simple risk-free rates for different times to maturity as of our valuation date, October 29, 2003. A line is fit to the data and its formula given.



Which volatility measure to use: historical or implied?

Determination of the best volatility value to use in Black-Scholes or risk-neutral calculations is not a clear-cut science. A fundamental choice is whether to use the stock price's historical volatility or to use the volatility implied in current market prices of options on the stock. Historical and implied volatilities often have different values.

Historical volatility is a backward-looking measure.

Historical volatility is the volatility that a stock price has exhibited in the past. It can be thought of as a backward-looking measure. Historical measures of volatility themselves often give different values of volatility depending on how far back in history one reaches to obtain stock-price data. A common practice is to reach as far back into the past as the expiration date of the option being valued reaches into the future.

To calculate a stock's annualized historical volatility with spreadsheet software:

1. Obtain the stock's daily closing prices for a span of time equal to the option's time to expiration.
2. Beginning with the second day of data, find the natural log of each day's price divided by the previous day's price.
3. From the column of daily log returns, calculate the one-day standard deviation.
4. To annualize the daily standard deviation, multiply it by the square root of 252. (Volatility varies with the square root of time. A trading year has 252 days.)

To illustrate the technique, the table below calculates the annualized standard deviation from 28 days of XYZ's closing prices. The 28 days of data give an annualized standard deviation of 51.18%. (Data that reaches as far back into the past as the XYZ options' expiration reaches into the future— 2,596 calendar days, 1,793 trading days— give an annualized standard deviation of 76.1%.)

Date	Closing Price	Log Return*
9/19/2003	$15.75	
9/22/2003	$15.38	-0.02377
9/23/2003	$15.37	-0.00065
9/24/2003	$14.49	-0.05896
9/25/2003	$14.06	-0.03012
9/26/2003	$13.17	-0.06539
9/29/2003	$13.58	0.03066
9/30/2003	$13.73	0.01099
10/1/2003	$13.59	-0.01025
10/2/2003	$14.47	0.06274
10/3/2003	$14.48	0.00069
10/6/2003	$14.30	-0.01251
10/7/2003	$14.74	0.03031
10/8/2003	$14.39	-0.02403
10/9/2003	$15.25	0.05805
10/10/2003	$15.35	0.00654
10/13/2003	$15.47	0.00779
10/14/2003	$15.28	-0.01236
10/15/2003	$14.59	-0.04621
10/16/2003	$14.42	-0.01172
10/17/2003	$14.00	-0.02956
10/20/2003	$13.90	-0.00717
10/21/2003	$14.29	0.02767
10/22/2003	$13.65	-0.04582
10/23/2003	$13.57	-0.00588
10/24/2003	$14.09	0.03760
10/27/2003	$13.80	-0.02080
10/28/2003	$13.90	0.00722

One-Day Standard Deviation	Annualized Standard Deviation
0.0322	0.5118
3.22%	51.18%

* Log return = ln(Current Price/Previous Price)

Implied volatility gives us option values consistent with the market's expectations about a stock price's future volatility.

Implied volatility is the stock-price volatility that market prices of options on the stock imply. That is, if we know a stock's expected volatility, then we can calculate the option's value. Going in the other direction, if we know an option's market price, we can extract from the price the stock volatility that the price implies.

Implied volatility can be thought of as a forward-looking measure. It captures market participants' consensus expectations about future volatility of the stock. Our goal is to come up with option values that are consistent with prices of market-traded options. Using implied volatilities in our calculations gives such values.

To extract implied volatilities from option prices, implied-volatility calculators require the following information:

- Whether the option is a call or put
- Whether the option is American-style or European-style
- Market price of stock
- Strike price of option
- Market price of option
- Option's time to expiration
- Applicable continuously compounded risk-free rate
- Expected future stock dividends, if any

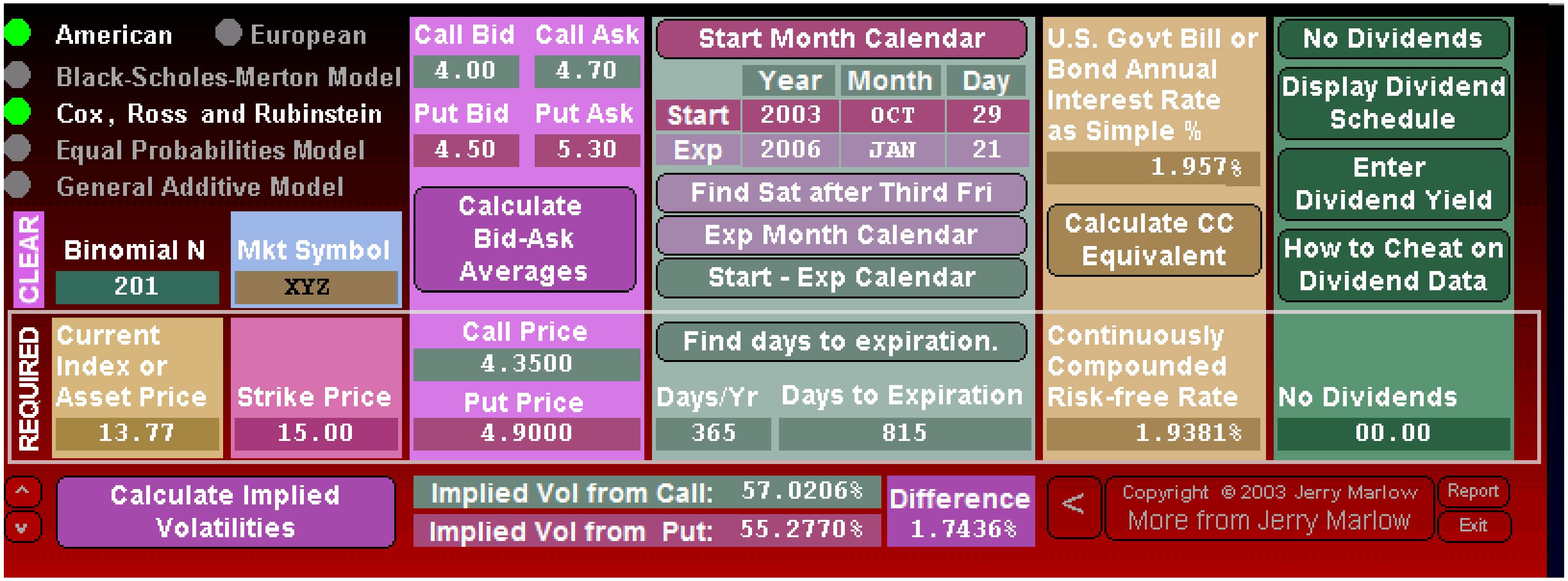
Given this information, the calculator systematically uses different volatility estimates to calculate an option value. When it calculates an option value that is sufficiently close to the market price of the option, the calculator displays the volatility it used as the implied volatility.

For example, for an October 29, 2003 as-of-valuation date, market data on options show an option on XYZ stock with these characteristics:

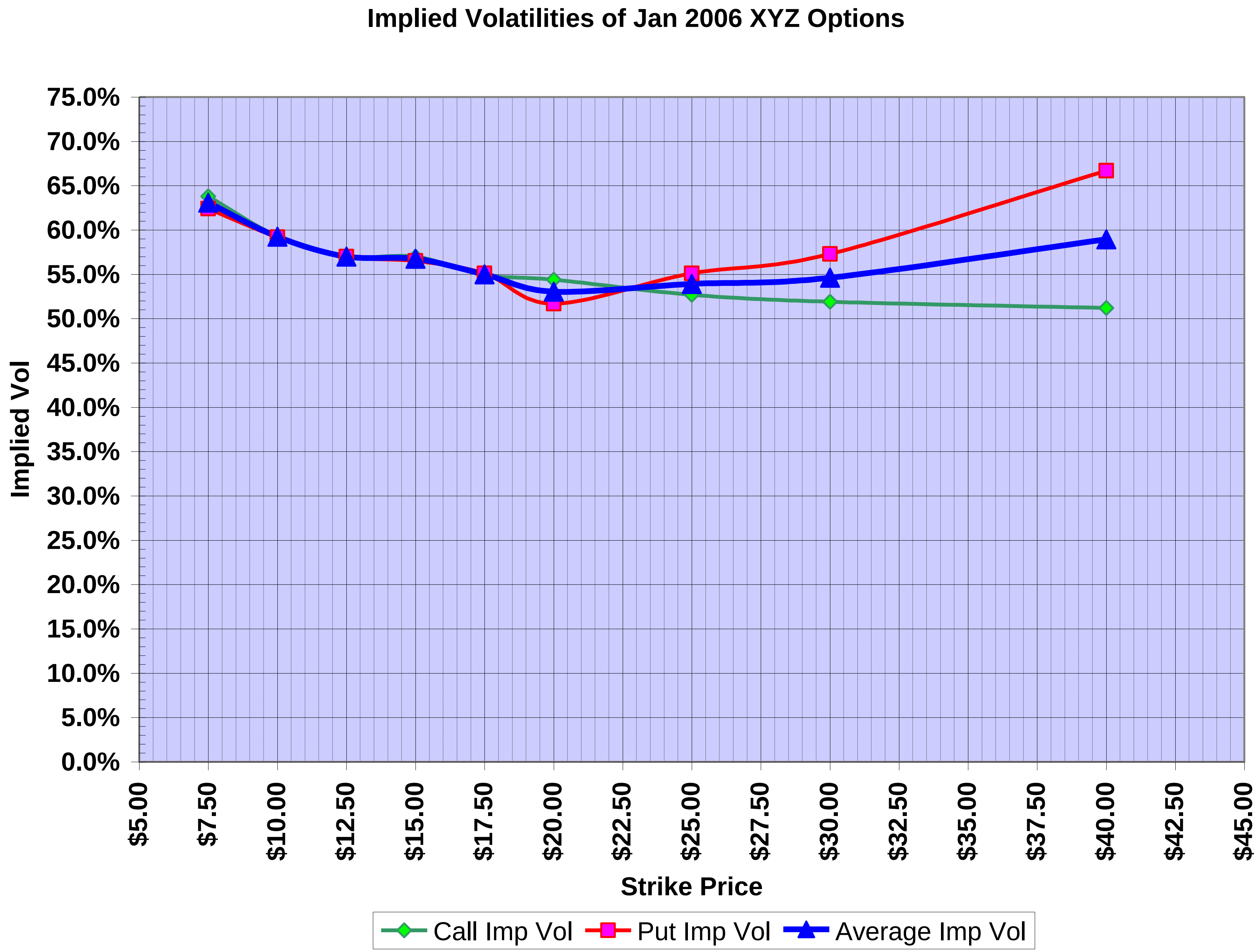
- Call
- American-style
- $13.77 market price of stock
- $15.00 strike price
- $4.00 bid / $4.70 ask price for option
- January 2006 expiration
- 1.9381% applicable continuously compounded risk-free rate
- The stock pays no dividends.

As an easy error-check on calculations, one can calculate implied volatility both from a call and the corresponding put. If market prices conformed exactly to theory, the call and the put would give exactly the same implied volatilities. If, in the actual calculations, the difference between the two calculated volatilities is large, then likely there is an error in the calculations.

For the above call, the corresponding put has a bid price of $4.50 and an ask of $5.30.

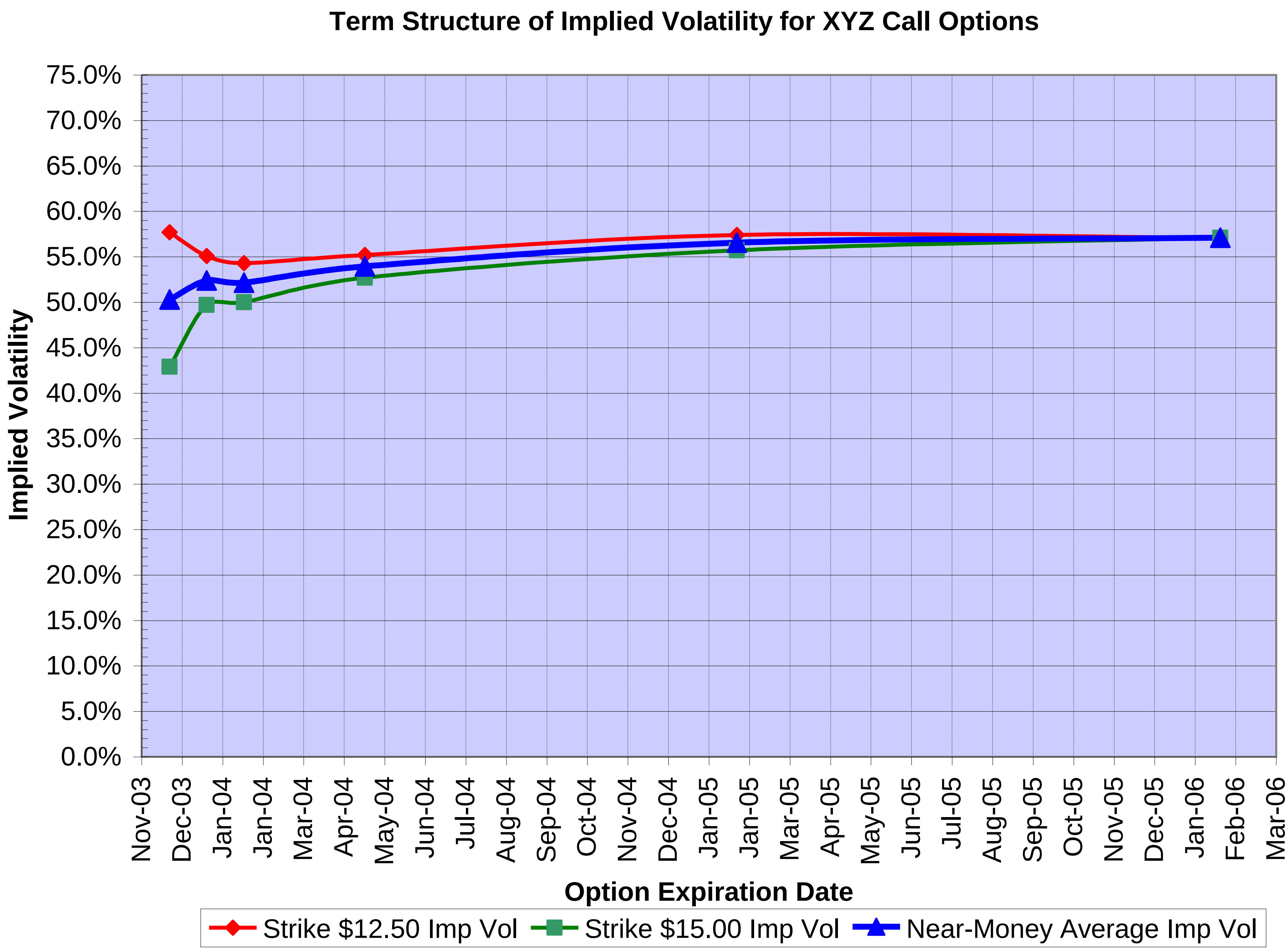


From the average of the call bid and ask prices, the calculator returns an implied volatility of 57.0206%. From the average of the put bid and ask prices, it returns an implied volatility of 55.2770%. A binomial N of 201 was chosen because, with that value, the Cox, Ross and Rubinstein model, from the call price, gives an implied volatility very near the value that the Black-Scholes-Merton model gives: 57.0340%.


Implied Volatilities of Jan 2006 XYZ Options
Implied Vol
75.0%
70.0%
65.0%
60.0%
55.0%
50.0%
45.0%
40.0%
35.0%
30.0%
25.0%
20.0%
15.0%
10.0%
5.0%
0.0%
$5.00
$7.50
$10.00
$12.50
$15.00
$17.50
$20.00
$22.50
$25.00
$27.50
$30.00
$32.50
$35.00
$37.50
$40.00
$42.50
$45.00
Strike Price
Call Imp Vol
Put Imp Vol
Average Imp Vol

The volatility smile

Calculating implied volatility from current option prices often yields a range of values. Options with the same time-to-expiration but different strike prices often yield a range of implied volatilities in which the implied volatilities of options with strike prices near the current stock price are lower than the implied volatilities of options with strike prices above or below the current stock price. A plot of values that has this pattern of relationships is known as “the volatility smile.”



Term Structure of Implied Volatility for XYZ Call Options
Implied Volatility
75.0%
70.0%
65.0%
60.0%
55.0%
50.0%
45.0%
40.0%
35.0%
30.0%
25.0%
20.0%
15.0%
10.0%
5.0%
0.0%
Nov-03
Dec-03
Jan-04
Jan-04
Mar-04
Apr-04
May-04
Jun-04
Jul-04
Aug-04
Sep-04
Oct-04
Nov-04
Dec-04
Jan-05
Jan-05
Mar-05
Apr-05
May-05
Jun-05
Jul-05
Aug-05
Sep-05
Oct-05
Nov-05
Dec-05
Jan-06
Feb-06
Mar-06
Option Expiration Date
Strike $12.50 Imp Vol
Strike $15.00 Imp Vol
Near-Money Average Imp Vol

The term structure of implied volatility

Options with the same strike price but different times-to-expiration often yield a range of values. The implied volatilities of options with near expirations may be higher or lower than those for distant expirations. A plot of these values reveals "the term structure of implied volatility" for the option chain.

Current market prices of options give different implied volatilities because market prices do not conform exactly with the assumptions that underlie Black-Scholes or standard binomial options pricing theory. The financial markets are auctions in which participants bid for and make offers to sell financial instruments. In the stock markets, estimates of firms' future values and/or earnings are auctioned. In the options markets, estimates of stocks' future volatility are auctioned. In both cases, market prices reflect the price level at which bids and offers clear the market. The bid- and offer-behavior of market participants does not conform exactly with the Black-Scholes or binomial pricing assumptions; hence, market prices of options do not conform with the models consistently across all strike prices and times to expiration.

Selection of a volatility value with which to value the options requires some exercise of judgment.

Calculating implied volatilities from options that have different strike prices and different times to expiration gives more than one implied volatility. Nonetheless, ranges are defined. Patterns and trends emerge. From these, the expert witness and the court can arrive at a volatility estimate in which they have confidence.

In the XYZ example, the term structure of the implied volatility increasingly flattens through the final data point of January 2006 at which point the implied volatility for the two nearest-the-money options is 57.1%. The average of the implied volatilities for the options with the most distant expiration date, January 2006, is 56.8%. To value the XYZ options at issue here, as the fairest available volatility measure, the expert witness elected to use 57%.

Calculation of option value if the options are vested and have a clear and clean expiration date.

If the options are vested and the option grant gives the employee an unambiguous expiration date for the options, then we have all the information we need to value the options. We can proceed to calculate the option's value.

If the options can be exercised at any time up until their expiration as is usually the case, then they are American-style.

The applicable risk-free rate can be calculated using the formula for the yield-curve.

In our XYZ example, we now have the following information:

Market price of stock as of the valuation date	$13.77
Strike price of option	$14.00
Valuation date	Oct 29, 2003
Expiration date of options	Dec 7, 2010
Applicable continuously compounded risk-free rate	3.5946%
Expected future volatility of stock	57%
Expected future stock dividends	None
Number of options	5,000

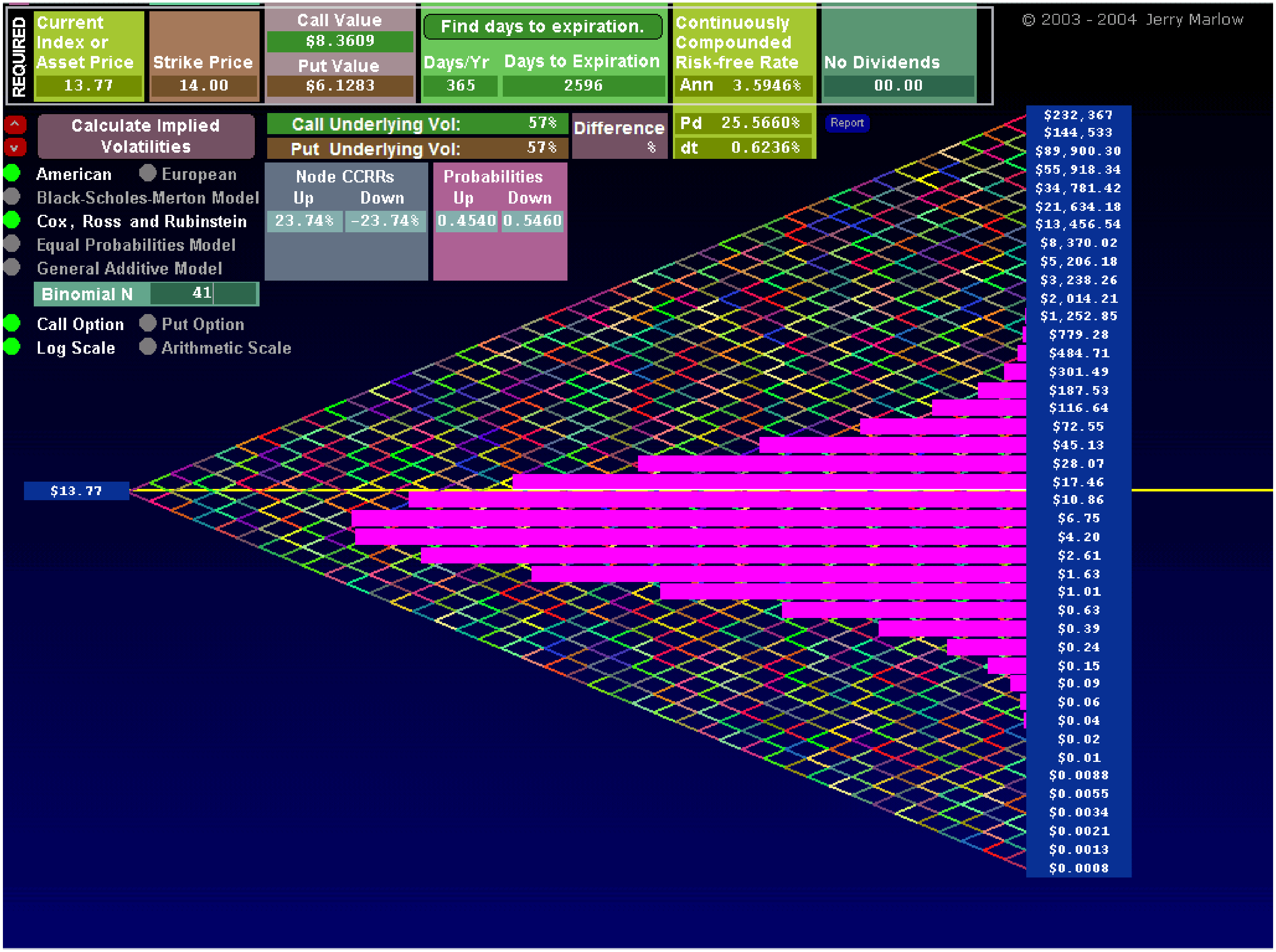
With a binomial N of 201, the Cox, Ross and Rubinstein model gives an option value of $8.3402. The value of 5,000 options is $41,701.

The other models give these values:

Equal Probabilities Model	$8.3049
General Additive Model	$8.3589
Black-Scholes Merton	$8.3352
Geometric-Brownian-Motion Model with 1,000,000 samples	$8.3382

If the options can be exercised at any time up until and at expiration, then they have the characteristics of an American-style option. The Black-Scholes-Merton model is designed for European-style options. Since the options are call options on an underlying that pays no dividends, the right of early exercise adds no value to them. Hence, the Black-Scholes-Merton model can be use to value them accurately.

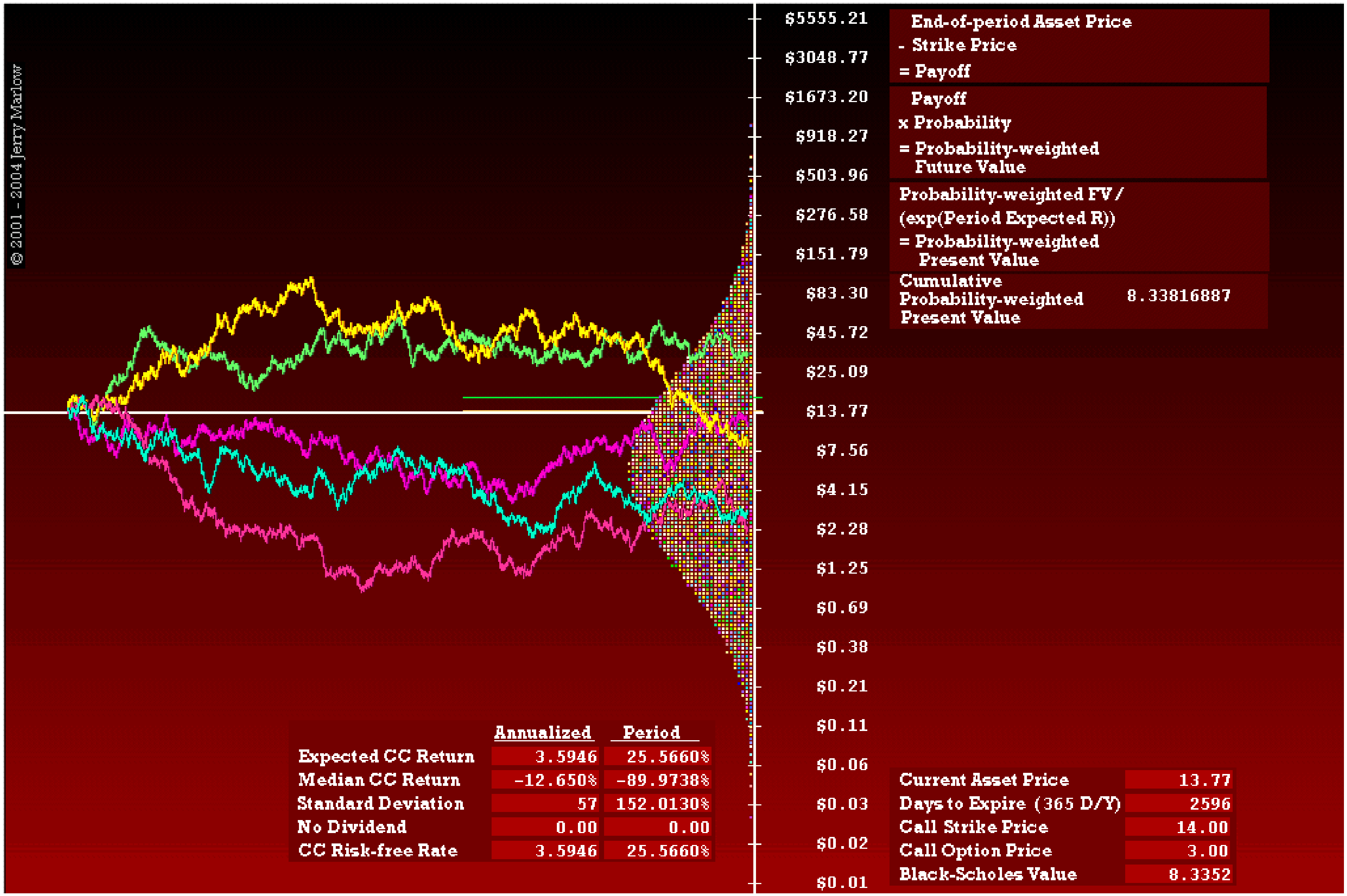
The value from the Cox, Ross, Rubinstein model is used because it was the model used to calculate the implied volatilities.


REQUIRED
Current Index or Asset Price
13.77
Strike Price
14.00
Call Value
$8.3609
Put Value
$6.1283
Find days to expiration.
Days/Yr
365
Days to Expiration
2596
Continuously Compounded Risk-free Rate
Ann 3.5946%
No Dividends
00.00
© 2003 - 2004 Jerry Marlow
Calculate Implied Volatilities
Call Underlying Vol: 57%
Put Underlying Vol: 57%
Difference
%
Pd 25.5660%
dt 0.6236%
Report
American
European
Black-Scholes-Merton Model
Cox, Ross and Rubinstein
Equal Probabilities Model
General Additive Model
Binomial N 41
Node CCRRs
Up Down
23.74% -23.74%
Probabilities
Up Down
0.4540 0.5460
Call Option
Put Option
Log Scale
Arithmetic Scale
$13.77
$232,367
$144,533
$89,900.30
$55,918.34
$34,781.42
$21,634.18
$13,456.54
$8,370.02
$5,206.18
$3,238.26
$2,014.21
$1,252.85
$779.28
$484.71
$301.49
$187.53
$116.64
$72.55
$45.13
$28.07
$17.46
$10.86
$6.75
$4.20
$2.61
$1.63
$1.01
$0.63
$0.39
$0.24
$0.15
$0.09
$0.06
$0.04
$0.02
$0.01
$0.0088
$0.0055
$0.0034
$0.0021
$0.0013
$0.0008

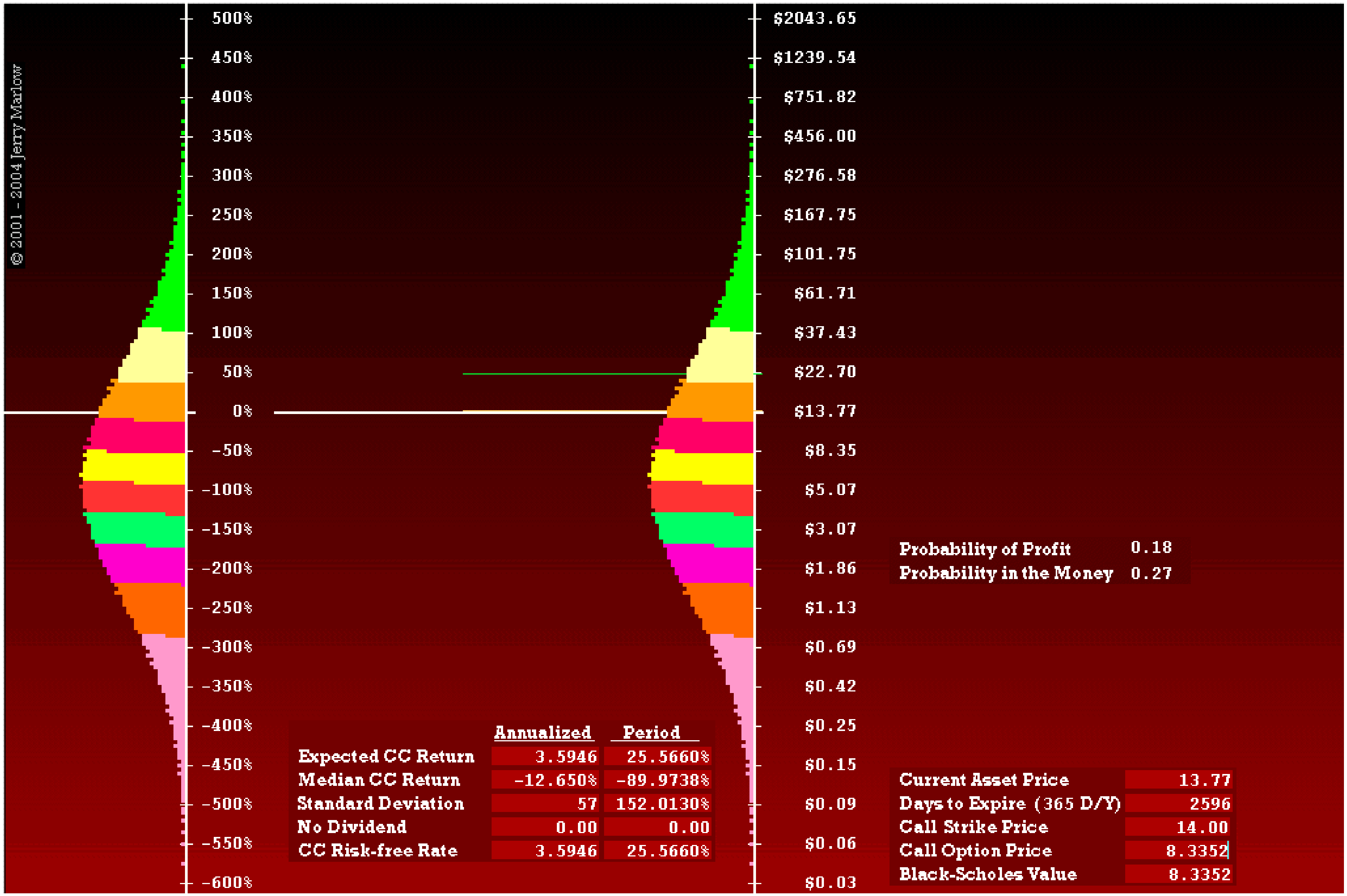
Probability distributions and simulations show the financial forecast and expectations on which the value is based.

If, in presenting an options value to a court, an expert witness provides only the number and the calculations by which the value was derived, the court either has to delve into the calculations or accept the assessment of the expert. Alternatively, an expert witness can provide the court with probability distributions that show the financial forecast and the expectations on which the value is based. He or she can show the court representative simulations. The court can see where the option value comes from.

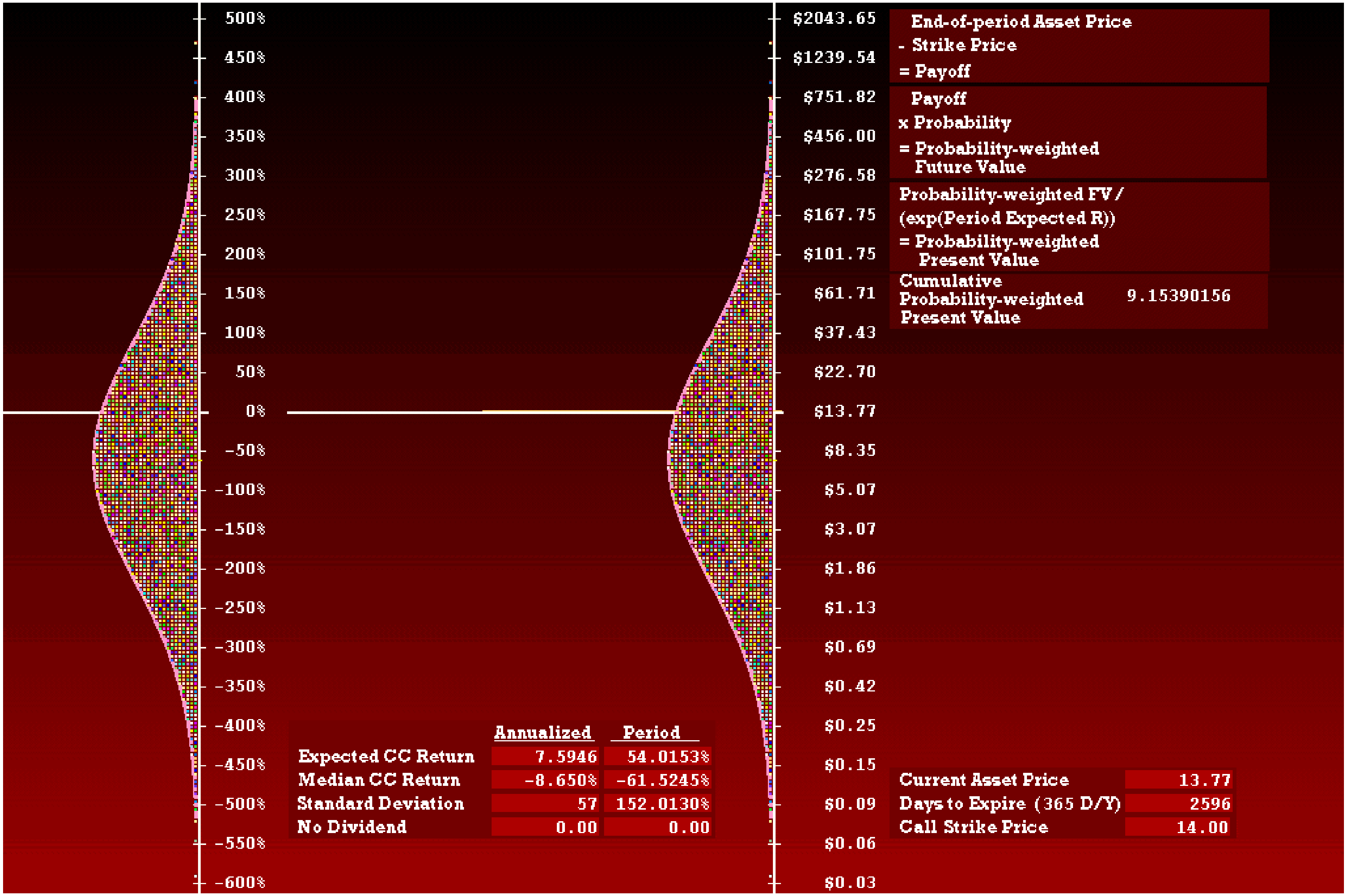
Though drawn with a lower binomial N which gives a slightly higher option value, the binomial distribution above shows that the value of the XYZ options is not based on an expectation that the company's stock is going to rise in value. The valuation forecast assumes that, on average, the stock is not expected to perform any better or any worse than a risk-free U.S. government bond. Even so, given how much volatility the financial markets expect the stock's price to have, there is some chance that the value of the stock may rise and even may rise significantly. The valuation process assesses how much the price of the stock might rise and the probabilities of those price rises occurring. It values the option accordingly.



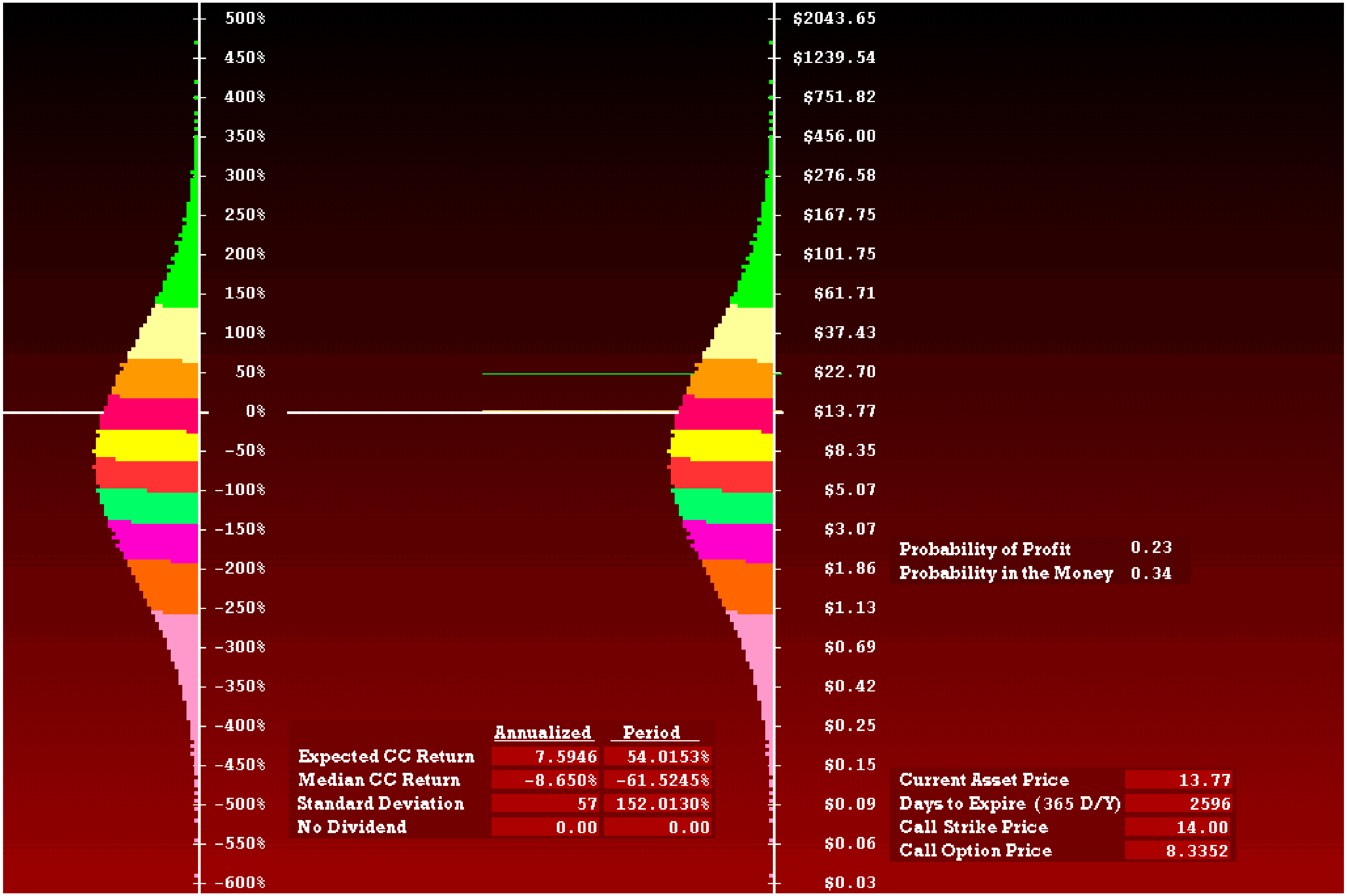
The geometric-Brownian-motion model shows the court that the valuation process does not rely on just a few hypothetical or conjectural payoffs that the option *might* provide. Rather the valuation process takes into account ***all*** the likely payoffs that the option might provide to its owner according to the consensus views reflected in option prices on the financial markets.



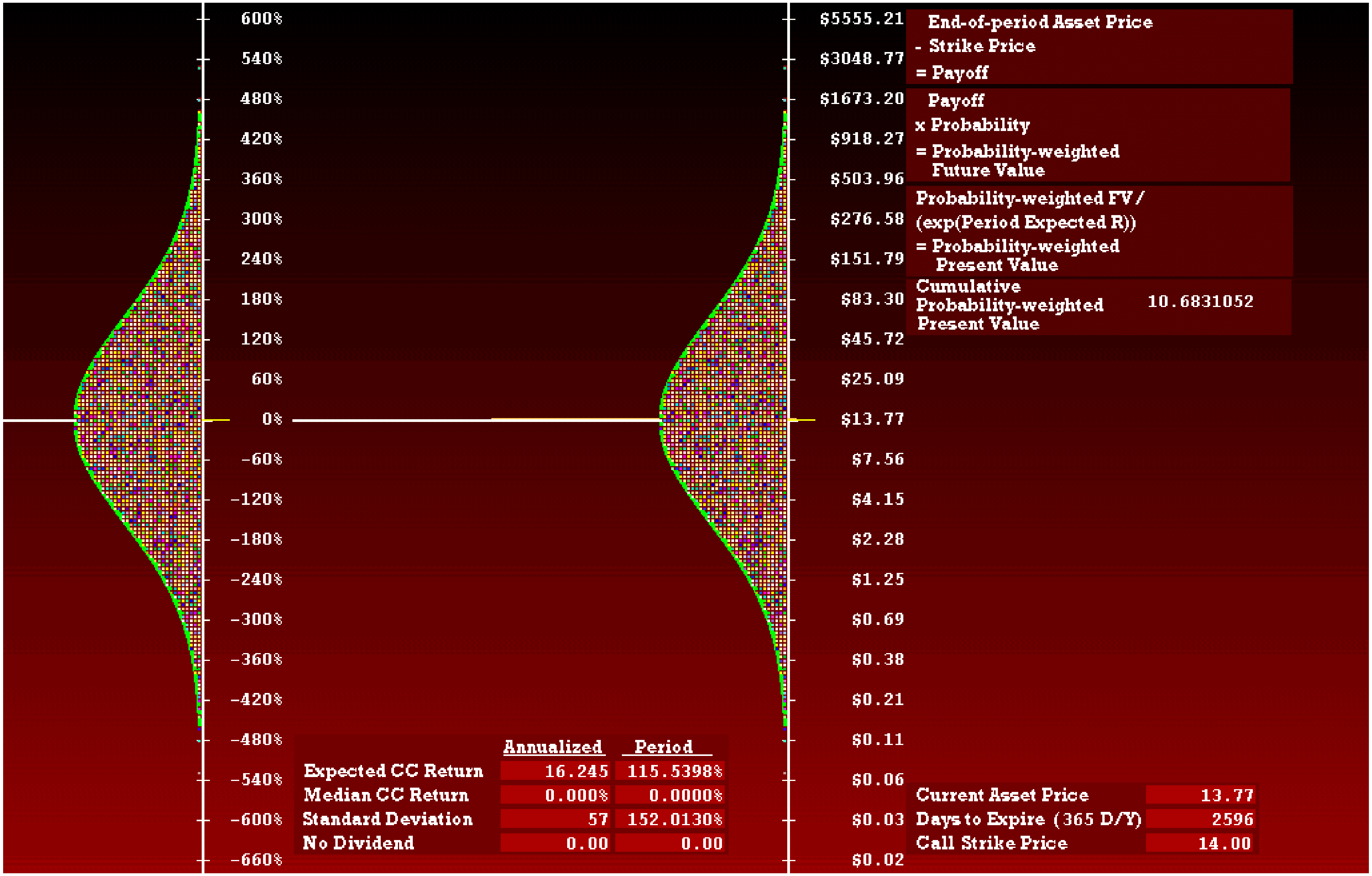
Under the valuation process's assumptions, probability distributions can show the court exactly how likely the option is to provide a payoff to the option owner. The valuation process assumes that, on average, the underlying stock will perform no better than a U.S. government bond. Under this assumption, the probability that the owner of the XYZ option will get a positive payoff is 0.27.



If an employee spouse argues that the payoff of an option will depend upon his or her future efforts for which he or she should be rewarded, then the employee spouse may, in effect, be arguing that, due to his or her efforts, the stock has an expected return greater than the risk-free rate. If that is the case, then the probability-weighted present value of the option is greater than its risk-neutral value.



If the valuation process took the expected return of the XYZ stock to be not the risk-free rate of 3.5946% but 7.5946%, then the option's probability-weighted present value would go from $8.34 to $9.15. The probability of a positive payoff would go from 0.27 to 0.34.



Were the XYZ employee to go so far as to argue that, through his or her efforts, the price of XYZ's stock is just as likely to go up as down, then he or she is, in effect, arguing that the call options have a value of at least $10.68.

What if the options are not yet vested?

In the preceding section, we valued vested options that expire in seven years. The employee had clear and clean contractual rights in the options. The options' expiration date was not subject to change. In short, the options had the characteristics of market-traded options. We used valuation methodologies originally designed for market-traded options.

But what if these options with seven years to expiration are not yet vested? What if they require that the employee remain an employee for an additional eighteen months from the valuation as-of date before the employee secures contractual rights comparable to those conferred by ownership of market-traded options?

We've seen that risk-neutral valuation sets a stock's expected return equal to the risk-free rate and finds the probability-weighted present value of an option's possible payoffs. For options that are not yet vested, we can extend this probabilistic approach and multiply the options' value by the probability that the employee spouse will still be an employee when the options vest.

The most objective measure of the probability that an employee will still be employed at the end of a given period is the geometric average of the company's annual rate of employee retention.

For example, suppose that over the past three years, XYZ had this history of employee retention:

	Year 1	Year 2	Year 3
Employees at beginning of year	2,000	3,000	4,000
Employees at beginning of year still employed at end of year	1,800	2,400	3,800
Annual retention rate	.90	.80	.95

The geometric average rate of retention would be:

$$\text{GAvRR} = (.90 \times .80 \times .95)^{(1/3)}$$
$$= 0.8811$$

Note in passing that, if the arithmetic average is used, the math is akin to that of the investment advisor who tells you that, since the stock he recommended went up by 40% one year and down by 30% the next, you have an average annual return of 5%. Meanwhile your $100,000 investment went up by 40% to $140,000 and then down by 30% to $98,000, so that, over the two years, the 5% average annual return reduced the value of your investment by $2,000.

If the geometric average rate of retention is 0.8811, then the probability that the employee will still be an employee 18 months from the valuation date is:

Probability still employed 18 months hence $= 0.8811^{(18/12)}$

$= 0.8271$

To obtain the risk-neutral value of the unvested options, multiply the risk-neutral value of one option times the number of options times the probability that the employee will still be employed long enough for the options to vest.

Value of unvested options = Risk-neutral value x Number of options x Probability still employed

= $8.3402 x 5,000 x 0.8271

= $34,490.90

We now have a risk-neutral, probability-weighted present value for the options. The value compensates the employee spouse for his or her exposure to uncertainty with a leveraged risk premium. However, no longer can we claim that this is a Black-Scholes value or an arbitrage-free value.

Black-Scholes option pricing theory assumes that investors can hedge their risks in the financial markets. Employees cannot hedge the risk that they will lose their jobs. Hence, once employee stock options lose their resemblance to market-traded options, we no longer can describe their values as Black-Scholes values or arbitrage-free values. Rather, they are risk-neutral values based on the expectation that the underlying stock will exhibit the volatility reflected in the prices of market-traded options and that, on average, the expected return of the stock is no greater than or any less than the risk-free rate.

Adjusting the value of unvested options for the probability that they will become vested is fair if the company customarily enforces its stipulation that an employee loses his or her options if he or she leaves the company before the options vest. If the company customarily waives that stipulation and gives departing employees the options, the adjustment may not need to be made.

What if the options are vested but the grant requires that, upon leaving the company, the employee exercise the options within a specified amount of time?

Market-traded options offer a clear right of exercise and a clear, unchanging expiration date. The longer an option's time to expiration, the greater its value. If an option grant stipulates that, upon leaving the company's employ, the employee must exercise the options within some time period, say sixty days, then the options do not have a clear, unchanging expiration date. Their effective expiration date is the earlier of their stated expiration date or sixty days after the employee leaves the company, whenever that may be. How do you value options when their effective expiration date is uncertain?

Just as we use probabilistic methods to value options in the face of uncertainties about the underlying stock's future market price and whether unvested options will actually vest, we can use probabilistic methods to value options in the face of uncertain effective expiration dates.

In the previous section, we calculated a company's geometric average rate of employee retention and used it as a best estimate of whether a given employee would still be an employee one year hence. Using that measure, we can calculate the probability that the employee spouse's employment will terminate on any given day, week, month or year.

To continue our XYZ example, let's say the options are vested and expire in 7.1123 years, but, if employment is terminated, the employee spouse must exercise his or her options within 60 days of termination.

In the table below, we make calculations based on termination opportunities occurring only once a month. For greater precision, they could be based on weekly or daily opportunities for termination.

n	Years Hence	Time Until Exp	Prob Still Employed	Prob Of Termination This Period Of Yr	Prob Termination This Opp	Prob This is Effec Exp	Num Of Options	Prob Weighted Num of Options with This Effec Exp	Risk Neutral Value	PW Value Emp Options
1	0.0000	0.1643	1.0000	0.0105	0.0105	0.0105	5,000	52.4662	$1.1730	$61.54

The first row of the table addresses the possibility that employment is terminated today, i.e., 0.0000 years hence. If employment terminates today, the options have an effective time until expiration of 0.1643 years:

60 days/365.25 = 0.1643 years.

The probability of termination this period of the year is equal to 1.0 – (geometric average rate of retention raised to the power of (1 over the number of termination opportunities per year).

Prob Termination This Period = 1.0 – $GeoAvRR^{(1/TermOppsPerYr)}$

= 1.0 - $0.8811^{(1/12)}$

= 1.0 - 0.9895

= 0.0105

The probability of termination this opportunity is equal to the probability still employed times the probability of termination this period of the year.

Prob Termination This Opp = Prob Still Employed x Probability Terminated This Pd of Year

= 1.0 x 0.0105

= 0.0105

The probability that the current time until expiration is the effective time until expiration is the same as the probability of termination this opportunity. If the employee spouse terminates today, the effective time until expiration is 60 days or 0.1643 years.

The number of options in the grant is 5,000.

To get a probability weighted number of options with the current effective time until expiration, we multiply the probability that the current time until expiration is the effective time until expiration times the number of options in the grant.

Prob Weighted Num of Options with This Effec Exp = Prob This is Effec Exp X Num Options
= 0.0105 x 5,000
= 52.4662

The risk-neutral value of the option is the value it would have with the current time until expiration and with an expected return equal to the continuously compounded risk-free rate appropriate for that time until expiration. The risk-neutral value of this option with a time to expiration of 0.1643 years is $1.1730.

The probability-weighted value of the number of options with this effective time to expiration is the probability-weighted number of options times the risk-neutral value.

52.4662 x $1.1730 = $61.54

n	Years Hence	Time Until Exp	Prob Still Employed	Prob Of Termination This Period Of Yr	Prob Termination This Opp	Prob This is Effec Exp	Num Of Options	Prob Weighted Num of Options with This Effec Exp	Risk Neutral Value	PW Value Emp Options
2	0.0833	0.2476	0.9895	0.0105	0.0104	0.0104	5,000	51.9156	$1.4660	$76.11
3	0.1667	0.3309	0.9791	0.0105	0.0103	0.0103	5,000	51.3709	$1.7129	$87.99
4	0.2500	0.4143	0.9688	0.0105	0.0102	0.0102	5,000	50.8318	$1.9306	$98.14
5	0.3333	0.4976	0.9587	0.0105	0.0101	0.0101	5,000	50.2984	$2.1275	$107.01
6	0.4167	0.5809	0.9486	0.0105	0.0100	0.0100	5,000	49.7707	$2.3087	$114.91
7	0.5000	0.6643	0.9387	0.0105	0.0098	0.0098	5,000	49.2484	$2.4775	$122.01
8	0.5833	0.7476	0.9288	0.0105	0.0097	0.0097	5,000	48.7316	$2.6362	$128.46
9	0.6667	0.8309	0.9191	0.0105	0.0096	0.0096	5,000	48.2203	$2.7864	$134.36
10	0.7500	0.9143	0.9094	0.0105	0.0095	0.0095	5,000	47.7143	$2.9293	$139.77

In subsequent rows of the table, the only difference of note is that, at each termination opportunity, the probability of termination decreases because one can be terminated only if one is still employed, and there is some chance that one will have terminated previously.

n	Years Hence	Time Until Exp	Prob Still Employed	Prob Of Termination This Period Of Yr	Prob Termination This Opp	Prob This is Effec Exp	Num Of Options	Prob Weighted Num of Options with This Effec Exp	Risk Neutral Value	PW Value Emp Options
11	0.8333	0.9976	0.8999	0.0105	0.0094	0.0094	5,000	47.2136	$3.0661	$144.76
12	0.9167	1.0809	0.8904	0.0105	0.0093	0.0093	5,000	46.7182	$3.1973	$149.37
13	1.0000	1.1643	0.8811	0.0105	0.0092	0.0092	5,000	46.2280	$3.3238	$153.65
14	1.0833	1.2476	0.8719	0.0105	0.0091	0.0091	5,000	45.7429	$3.4458	$157.62
15	1.1667	1.3309	0.8627	0.0105	0.0091	0.0091	5,000	45.2629	$3.5640	$161.32
16	1.2500	1.4143	0.8537	0.0105	0.0090	0.0090	5,000	44.7879	$3.6786	$164.76
17	1.3333	1.4976	0.8447	0.0105	0.0089	0.0089	5,000	44.3180	$3.7900	$167.97
18	1.4167	1.5809	0.8358	0.0105	0.0088	0.0088	5,000	43.8529	$3.8984	$170.96
19	1.5000	1.6643	0.8271	0.0105	0.0087	0.0087	5,000	43.3928	$4.0041	$173.75
20	1.5833	1.7476	0.8184	0.0105	0.0086	0.0086	5,000	42.9374	$4.1072	$176.35
21	1.6667	1.8309	0.8098	0.0105	0.0085	0.0085	5,000	42.4869	$4.2079	$178.78
22	1.7500	1.9143	0.8013	0.0105	0.0084	0.0084	5,000	42.0411	$4.3065	$181.05
23	1.8333	1.9976	0.7929	0.0105	0.0083	0.0083	5,000	41.5999	$4.4029	$183.16
24	1.9167	2.0809	0.7846	0.0105	0.0082	0.0082	5,000	41.1634	$4.4974	$185.13
25	2.0000	2.1643	0.7763	0.0105	0.0081	0.0081	5,000	40.7315	$4.5901	$186.96
26	2.0833	2.2476	0.7682	0.0105	0.0081	0.0081	5,000	40.3040	$4.6810	$188.66
27	2.1667	2.3309	0.7601	0.0105	0.0080	0.0080	5,000	39.8811	$4.7702	$190.24
28	2.2500	2.4143	0.7522	0.0105	0.0079	0.0079	5,000	39.4626	$4.8579	$191.71
29	2.3333	2.4976	0.7443	0.0105	0.0078	0.0078	5,000	39.0486	$4.9441	$193.06
30	2.4167	2.5809	0.7365	0.0105	0.0077	0.0077	5,000	38.6388	$5.0288	$194.31
31	2.5000	2.6643	0.7287	0.0105	0.0076	0.0076	5,000	38.2334	$5.1122	$195.46
32	2.5833	2.7476	0.7211	0.0105	0.0076	0.0076	5,000	37.8322	$5.1943	$196.51
33	2.6667	2.8309	0.7135	0.0105	0.0075	0.0075	5,000	37.4352	$5.2751	$197.48
34	2.7500	2.9143	0.7060	0.0105	0.0074	0.0074	5,000	37.0424	$5.3547	$198.35
35	2.8333	2.9976	0.6986	0.0105	0.0073	0.0073	5,000	36.6537	$5.4332	$199.15
36	2.9167	3.0809	0.6913	0.0105	0.0073	0.0073	5,000	36.2691	$5.5105	$199.86
37	3.0000	3.1643	0.6840	0.0105	0.0072	0.0072	5,000	35.8885	$5.5868	$200.50
38	3.0833	3.2476	0.6769	0.0105	0.0071	0.0071	5,000	35.5119	$5.6620	$201.07
39	3.1667	3.3309	0.6698	0.0105	0.0070	0.0070	5,000	35.1393	$5.7361	$201.56
40	3.2500	3.4143	0.6627	0.0105	0.0070	0.0070	5,000	34.7705	$5.8093	$201.99
41	3.3333	3.4976	0.6558	0.0105	0.0069	0.0069	5,000	34.4057	$5.8816	$202.36
42	3.4167	3.5809	0.6489	0.0105	0.0068	0.0068	5,000	34.0447	$5.9529	$202.66
43	3.5000	3.6643	0.6421	0.0105	0.0067	0.0067	5,000	33.6874	$6.0233	$202.91

44	3.5833	3.7476	0.6353	0.0105	0.0067	0.0067	5,000	33.3339	$6.0928	$203.10
45	3.6667	3.8309	0.6287	0.0105	0.0066	0.0066	5,000	32.9841	$6.1615	$203.23
46	3.7500	3.9143	0.6221	0.0105	0.0065	0.0065	5,000	32.6380	$6.2293	$203.31
47	3.8333	3.9976	0.6155	0.0105	0.0065	0.0065	5,000	32.2956	$6.2964	$203.34
48	3.9167	4.0809	0.6091	0.0105	0.0064	0.0064	5,000	31.9567	$6.3626	$203.33
49	4.0000	4.1643	0.6027	0.0105	0.0063	0.0063	5,000	31.6213	$6.4280	$203.26
50	4.0833	4.2476	0.5964	0.0105	0.0063	0.0063	5,000	31.2895	$6.4927	$203.15
51	4.1667	4.3309	0.5901	0.0105	0.0062	0.0062	5,000	30.9612	$6.5567	$203.00
52	4.2500	4.4143	0.5839	0.0105	0.0061	0.0061	5,000	30.6363	$6.6199	$202.81
53	4.3333	4.4976	0.5778	0.0105	0.0061	0.0061	5,000	30.3148	$6.6824	$202.58
54	4.4167	4.5809	0.5717	0.0105	0.0060	0.0060	5,000	29.9967	$6.7442	$202.30
55	4.5000	4.6643	0.5657	0.0105	0.0059	0.0059	5,000	29.6820	$6.8053	$201.99
56	4.5833	4.7476	0.5598	0.0105	0.0059	0.0059	5,000	29.3705	$6.8657	$201.65
57	4.6667	4.8309	0.5539	0.0105	0.0058	0.0058	5,000	29.0623	$6.9254	$201.27
58	4.7500	4.9143	0.5481	0.0105	0.0058	0.0058	5,000	28.7574	$6.9845	$200.86
59	4.8333	4.9976	0.5424	0.0105	0.0057	0.0057	5,000	28.4556	$7.0430	$200.41
60	4.9167	5.0809	0.5367	0.0105	0.0056	0.0056	5,000	28.1570	$7.1008	$199.94
61	5.0000	5.1643	0.5310	0.0105	0.0056	0.0056	5,000	27.8616	$7.1580	$199.43
62	5.0833	5.2476	0.5255	0.0105	0.0055	0.0055	5,000	27.5692	$7.2146	$198.90
63	5.1667	5.3309	0.5200	0.0105	0.0055	0.0055	5,000	27.2799	$7.2706	$198.34
64	5.2500	5.4143	0.5145	0.0105	0.0054	0.0054	5,000	26.9937	$7.3260	$197.75
65	5.3333	5.4976	0.5091	0.0105	0.0053	0.0053	5,000	26.7104	$7.3808	$197.14
66	5.4167	5.5809	0.5038	0.0105	0.0053	0.0053	5,000	26.4301	$7.4350	$196.51
67	5.5000	5.6643	0.4985	0.0105	0.0052	0.0052	5,000	26.1528	$7.4886	$195.85
68	5.5833	5.7476	0.4932	0.0105	0.0052	0.0052	5,000	25.8784	$7.5417	$195.17
69	5.6667	5.8309	0.4881	0.0105	0.0051	0.0051	5,000	25.6068	$7.5942	$194.46
70	5.7500	5.9143	0.4829	0.0105	0.0051	0.0051	5,000	25.3381	$7.6462	$193.74
71	5.8333	5.9976	0.4779	0.0105	0.0050	0.0050	5,000	25.0722	$7.6976	$193.00
72	5.9167	6.0809	0.4729	0.0105	0.0050	0.0050	5,000	24.8092	$7.7485	$192.23
73	6.0000	6.1643	0.4679	0.0105	0.0049	0.0049	5,000	24.5488	$7.7988	$191.45
74	6.0833	6.2476	0.4630	0.0105	0.0049	0.0049	5,000	24.2912	$7.8486	$190.65
75	6.1667	6.3309	0.4581	0.0105	0.0048	0.0048	5,000	24.0363	$7.8979	$189.84
76	6.2500	6.4143	0.4533	0.0105	0.0048	0.0048	5,000	23.7841	$7.9467	$189.01
77	6.3333	6.4976	0.4486	0.0105	0.0047	0.0047	5,000	23.5345	$7.9950	$188.16
78	6.4167	6.5809	0.4439	0.0105	0.0047	0.0047	5,000	23.2876	$8.0428	$187.30
79	6.5000	6.6643	0.4392	0.0105	0.0046	0.0046	5,000	23.0432	$8.0901	$186.42
80	6.5833	6.7476	0.4346	0.0105	0.0046	0.0046	5,000	22.8014	$8.1368	$185.53
81	6.6667	6.8309	0.4300	0.0105	0.0045	0.0045	5,000	22.5622	$8.1831	$184.63
82	6.7500	6.9143	0.4255	0.0105	0.0045	0.0045	5,000	22.3254	$8.2289	$183.71
83	6.8333	6.9976	0.4211	0.0105	0.0044	0.0044	5,000	22.0912	$8.2743	$182.79

84	6.9167	7.0809	0.4166	0.0105	0.0044	0.0044	5,000	21.8593	$8.3191	$181.85
85	7.0000	7.1123	0.4123	0.0105	0.0043	0.0043	5,000	21.6300	$8.3380	$180.35
86	7.0833	7.1123	0.4079	0.0105	0.0043	0.0043	5,000	21.4030	$8.3414	$178.53

The calculation of probabilities of termination and effective time to expiration continues until we reach the point in time when the options expire according to their expiration date.

	Time Until Grant Exp	Time Until Exp	Prob Still Employed		Prob Still Employed	Prob This is Effec Exp	Num Of Options	Prob Weighted Num of Options with This Effec Exp	Risk Neutral Value	PW Value Emp Options
	7.1123	7.1123	0.4037		0.4037	0.4037	5,000	2,018.2918	$8.3414	$16,835.31

At the point in time when the options' stated expiration date is reached, the employee has some probability of still being employed. For that probability, the options' effective time to expiration is the grant's stated time to expiration. The options' risk-neutral value is their value for the full time to expiration.

			Cumulative Prob Either Emp Or Terminated				Total Num of Prob Weighted Options		Cumulative PW Value Emp Options
			1.0000				5,000		$32,391.33

As checks to the calculations, the "probabilities of termination this opportunity" and the probability of still being employed at the grant's stated expiration date should add up to 1.0. The "probability-weighted number of options with this expiration date" should add up to the number of options in the grant.

The cumulative probability-weighted value of the options in the grant is the sum of the column labeled "Probability-weighted value employee options."

The cumulative probability-weighted value of the 5,000 XYZ options is $32,391.33.

If the options were not yet vested *and* stipulated that, upon termination, the employee must exercise them within a specified time, then we could combine the methodology of this section with that of the previous section.

“Intrinsic value”

Nowhere in our discussion of option values have we mentioned “intrinsic value.” An option’s so-called intrinsic value is the value the option would have if it expired on the valuation date. The longer an option’s time to expiration, the greater its value. If an option expires anytime after its valuation date, the expression “intrinsic value” is irrelevant and potentially misleading. All the options used as examples in this paper have an “intrinsic value” of zero. To make sense of the expression “intrinsic value,” always replace it with the language “expire-today value.”

Send the author your thoughts

The seminar and its learning materials are undergoing constant enrichment. Send seminar leader Jerry Marlow your comments, criticisms, suggestions and inquiries. Let him know how he can make this paper more useful to courts, attorneys, valuation professionals and divorce financial planners.

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Jerry Marlow
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